Lecture 20

Related Rates

Ex:

Balloon (spherical)

Air flowing in at \(1 \frac{m^3}{s}\)

Q: What is the rate of change of the radius of the balloon? Assume \(r = 10\).

SOL: Equation linking \(r\) with volume of balloon

\[ V = \frac{4}{3} \pi r^3 \]
Express "air flows in at $1 \frac{m^3}{s}\) as a derivative:

$$\frac{dV}{dt} = 1 \left(\frac{m^3}{s}\right)$$

Usually omit unit.

What we need:

$$\frac{dr}{dt}$$

Rate of change of radius over time.

Use the chain rule:

$$V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3} \pi \cdot 3r^2 \cdot \frac{dr}{dt}$$
(Alternatively, think of implicit differentiation)

\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}
\]

Solve for this:

\[
\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt}
\]

\[
\frac{dV}{dt} = 1
\]

\[r = 10 \text{ (from question)}\]

So, when \( r = 10 \),

\[
\frac{dr}{dt} = 4\pi \cdot 10^2 \cdot 1 = \frac{1}{400\pi}
\]
So radius changes at \( t \) 
\[
\frac{1}{4000 \pi \text{ (m/s)}}.
\]

Derivative step

\[ V = \frac{4}{3} \pi r^3 \]

Idea: \( V \) is a function of \( t \)
\( (t = \text{time}) \)

\[
\frac{dV}{dt} = \frac{d}{dt} \left( \frac{4}{3} \pi r^3 \right)
\]

\[
= \frac{4}{3} \pi \frac{dr}{dt} (r^3)
\]

\[
= \frac{4}{3} \pi \frac{3r^2}{dr/dt} \frac{dr}{dt}
\]

Outer function derivative

Inner function derivative
\[
\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} \quad (\text{Chain Rule})
\]

\[V = \frac{4}{3} \pi r^3\]

\[\frac{dV}{dr} = 4 \pi r^2\]

So:
\[\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt}\]

\[x^2 + y^2 = 1\]

\[2x + 2y \cdot \frac{dy}{dx} = 0\]

Implicit

Differentiation
Ex: Plane A travels due N from airport, Plane B travels due E, both constant altitude. What is the rate of change of distance between A and B, when:

- A is 400 km from airport
- B is 300 km from airport
- both travel at $800 \frac{\text{km}}{\text{h}}$

Sol:

Diagram:

- A
- B
- Distance between A and B

Airport 400 300
Equation linking \( R, x, y, z \): 

\[ x^2 + y^2 + z^2 = R^2 \]

\[ \frac{dx}{dt} = 300 \]

\[ y = 400 \]

\[ z = 0 \]

What are we given:

\[ \text{distance} \]

\[ \text{air port} \]

\[ \frac{dx}{dt} = \frac{d}{dt} \]
Use Implicit Differentiation to find \( \frac{dx}{dt} \):

\[
\frac{d}{dt} (x^2) = \frac{d}{dt} (x^2 + z^2)
\]

\[
2x \frac{dx}{dt} = 2y \frac{dy}{dt} + 2z \frac{dz}{dt}
\]

Solve for this:

\[
\frac{dx}{dt} = \frac{0 - y \frac{dy}{dt} - z \frac{dz}{dt}}{x}
\]

Need to find \( x \):

\[
x^2 = y^2 + z^2
\]

\[
x^2 = 400^2 + 300^2 = 250000
\]

\[
x = 500
\]
\[
\frac{dx}{dt} = \frac{400 \cdot 800 + 300 \cdot 800}{500}
\]
\[
= 800 \cdot \frac{700}{500}
\]
\[
= 900 \cdot \frac{7}{5}
\]
\[
= 160 \cdot 7
\]
\[
\frac{dx}{dt} = \Delta \text{km} 1120
\]

So distance between planes changes at 1120 km/h.

"Check" your answer:

\[
\begin{align*}
800 \quad \text{km} \\
\quad \text{ht}
\end{align*}
\]

\[
\text{Lo} \cdot \text{yo} \quad \text{seems reasonable, also 0 > 800.}
\]
Ex: Street light at 8 m height, 1.6 m tall woman walking towards light at 1 m/s. What is rate of change of length of her shadow?

SoC:

Given: \( \frac{dy}{dt} = -1 \text{ (m/s)} \)

Want: \( \frac{dx}{dt} \)
Equation relating to $x, y$:

\[
\frac{8}{x+y} = \frac{1.6}{x}
\]

Solve for $x$:

\[
8x = 1.6(x + y)
\]

\[
6.4x = 1.6y
\]

\[
x = \frac{y}{4}
\]

\[
\frac{dx}{dt} = \frac{1}{4} \frac{dy}{dt}
\]

\[
= -\frac{1}{4} \left( \frac{m}{s} \right)
\]

So: length of shadow decreases at

\[
\frac{1}{4} \frac{m}{s}
\]