Lecture 19

Compound Interest

Recall:

\( PV \): Present Value

\( FV \): Future Value

\( r \): Interest rate

\( t \): Number of years

\( n \): Compounding frequency

\[
FV = PV \cdot \left(1 + \frac{r}{n}\right)^{nt}
\]
Ex: $1000 compounded semi-annually at 6% for 5 years. How much do we have after 5 years?

So C: \[ PV = 1000 \]
\[ r = 0.06 \]
\[ t = 5 \]
\[ n = 2 \]

\[ FV = 1000 \left( 1 + \frac{0.06}{2} \right)^{2 \cdot 5} \]
\[ = 1000 \left( 1 + 0.03 \right)^{10} \]
\[ \approx 1343.9 \]
Could imagine $n \to \infty$:

**Continuous Compounding:**

Ex: $1$ cent is compound over 1 year at 10% interest.

So: $\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n$

This will be $n$-times compounded with interest $r$.

So: $\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e^r$

( Difficult to prove, use it as a fact)
Here: at 1% after 1 year,

\[ FV = e^{0.01} \]

In general:

\[ FV = PV \cdot e^{rt} \]

- \( r \): interest rate
- \( t \): time in years

Continuous Compounding
Q11

\[ FV = 5000 \]
\[ \epsilon = 2 \]
\[ r = 0.12 \]

\[ FV = PV e^{rt} \]
\[ 5000 = PV e^{2 \times 0.12} = PV e^{0.24} \]
\[ PV = \frac{5000}{e^{0.24}} = \frac{5000}{e^{0.24}} \]

Q31

\[ FV = 2500 \]
\[ PV = 1000 \]
\[ r = 0.06 \]

\[ 2500 = 1000 \times e^{0.06} \]
2.5 = e^{0.06t}

\ln(2.5) = 0.06t

\[ t = \frac{\ln(2.5)}{0.06} = 15 \]

Q41 \quad FV = PV \cdot e^{rt}

Property value triples between 2001 - 2011:

3PV = PV \cdot e^{r \cdot 10}

3 = e^{r \cdot 10}

\ln(3) = 10r

\[ r = \frac{\ln(3)}{10} \]
\[ FV = PV e^{rt} \]
\[ 5 \cdot PV = PV e^{\frac{ln(3)}{10} \cdot t} \]
\[ 5 = e^{\frac{ln(3)}{10} \cdot t} \]
\[ ln(5) = \frac{ln(3)}{10} \cdot t \]
\[ t = 10 \cdot \frac{ln(5)}{ln(3)} \approx 4.6 \]

20 \approx 2015.6

Q6 \]
\[ PV_A = 870000 \]
\[ r_A = 0.13 \]
\[ PV_B = 600000 \]
\[ r_B = 0.14 \]
\[ FV_A = PV_A \cdot e^{r_A \cdot t} \]
\[ FV_B = PV_B \cdot e^{r_B \cdot t} \]

So, want \( t \) such that
\[ PV_A \cdot e^{r_A \cdot t} = PV_B \cdot e^{r_B \cdot t} \]

\[ 70000 \cdot e^{0.13 \cdot t} = 60000 \cdot e^{0.14 \cdot t} \]

\[ 70000 = 60000 \cdot e^{0.01 \cdot t} \]

\[ \frac{7}{6} = e^{0.01 \cdot t} \]

\[ \ln\left(\frac{7}{6}\right) = 0.01 \cdot t \]

\[ t = 100 \cdot \ln\left(\frac{7}{6}\right) = 15.4 \]
Ex: Assume you have an investment that generates $1000/month when compounded continuously at 5% interest/year for all eternity.

How large does the investment need to be?

SOC: \[ PV + 1000 = PV e^{0.05 \times \frac{1}{12}} = FV \]

This is how much we get after 1 month.
Solve this for PV:

\[ PV \left(1 - e^{0.05 \cdot \frac{1}{12}}\right) = -1000 \]

\[ PV \left(e^{0.05 \cdot \frac{1}{12}} - 1\right) = 1000 \]

\[ PV = \frac{1000}{e^{0.05 \cdot \frac{1}{12}} - 1} \]

\[ \approx 239,500 \]

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**Exponential Models**

\[ Y(t) = C \cdot e^{K \cdot t} \]

- \( Y = 10 \cdot e^{10t} \)  \( \text{C: Constant} \)
- \( Y = 0.1 \cdot e^{0.1t} \)  \( \text{K: Growth rate} \)
\[
\frac{d}{dt} Y = (e^{k \cdot t} \cdot k = k \cdot Y
\]

Relative growth rate:

\[
\frac{dY}{dt} = k \cdot Y
\]

*Constant for any exponential model.*

Ex: World population
- 6 billion in 1995 (t=0)
- 6.9 billion in 2005 (t=10)

Q: Fit an exponential growth function to those values.
\[ y(t) : \text{population at time } t \text{ in billions} \]

\[ y(0) = 6 = C \cdot e^{k \cdot 0} = C \]

\[ y(10) = 6.9 = C \cdot e^{k \cdot 10} \]

So: \[ C = 6 \] so \[ 6.9 = 6 \cdot e^{10k} \]

\[ \frac{6.9}{6} = e^{10k} \]

\[ \ln \left( \frac{6.9}{6} \right) = 10k \]

\[ k = \frac{1}{10} \ln \left( \frac{6.9}{6} \right) \approx 0.014 \]

Q: How many years does it take for population to double?
$\text{SoC: } Y(t) = 2 \cdot y(0)$

$\chi e^{4t} = 2 \cdot \chi e^{4 \cdot 0}$

$e^{4t} = 2$

$\frac{1}{10} \ln \left( \frac{6.9}{6} \right) \cdot t$

$\chi = 2$

$\frac{1}{10} \ln \left( \frac{6.9}{6} \right) \cdot t = \ln(2)$

$t = \frac{10 \cdot \ln(2)}{\ln \left( \frac{6.9}{6} \right)}$

$t \approx 50$