Normal Lines

\[ f'(x) \]
\[ x=\alpha \]

**FACT:** If the slope of the tangent is \( m \), then the slope of the normal is \( -\frac{1}{m} \).
Ex: \[ x^2 + xy + y^2 = 7 \]

Q: Find the equation of the normal line at \((2, 1)\):

So C: Step 1 - Scope of tangent

Implicit Diff: \[ \frac{d}{dx}(x^2 + xy + y^2) = 0 \]

\[ 2x + x + y + 2y \left( \frac{dy}{dx} \right) = 0 \]

\[ \frac{dx}{dy} \left( x + 2y \right) = -2x-y \]

\[ \frac{dy}{dx} = \frac{-2x-y}{x+2y} \]
Set \( x = 2 \); \( y = 1 \):

\[
\frac{dy}{dx} = \frac{-4 - 1}{2 + 2} = \frac{-5}{4}
\]

Scope of tangent is \(-\frac{5}{4}\)

\(\rightarrow\) Scope of normal is \(-\left(\text{product of slopes}\right)\) = \(\frac{4}{5}\)

Scope \(\frac{4}{5}\)

Step 2: Equation of normal line
\[ N(x) = \frac{4}{5} x + n \]

Normal line

\[ N(2) = 1 \quad (\text{passes through } (2,1)) \]

\[ = \frac{4}{5} \cdot 2 + n \]

\[ n = \frac{8}{5} - \frac{3}{5} = \frac{-3}{5} \]

\[ N(x) = \frac{4}{5} x - \frac{3}{5} \]
Ex: \[ x^2 + 9y^2 = 9 \]

\[ P = \left(\frac{4}{3}, 0\right) \]

Q: For which points on the ellipse does the normal line pass through P?

Sol: Take \((a, b)\) on ellipse, want to find equation for normal line at \((a, b)\).
Ellipse: \[ x^2 + 9y^2 = 9 \]

Implicit Diff: \[ 2x + 18y \frac{dy}{dx} = 0 \]

\[
18y \frac{dy}{dx} = -2x
\]

\[
\frac{dy}{dx} = -\frac{2x}{18y} = -\frac{x}{9y}
\]

Slope of tangent at \((a, b)\):

\[
-\frac{a}{9b}
\]

Slope of normal at \((a, b)\):

\[
\frac{9b}{a}
\]
\[ N(x) = \frac{9b}{a} \cdot x + n \]

\[ N(a) = b = \frac{9b}{a} \cdot a + n = b \]

\[ n = b - 9b = -8b \]

So:

\[ N(x) = \frac{9b}{a} \cdot x - 8b \]

Now: want \( N(x) \) to pass through \( P = (\frac{4}{3}, 0) \)
\[ N \left( \frac{4}{3} \right) = 0 \]

\[ \frac{9b}{a} \cdot \frac{4}{3} - 8b = 0 \]

Solve:

\[ b \left( \frac{12}{a} - 8 \right) = 0 \]

Either: \( b = 0 \) or \( \frac{12}{a} - 8 = 0 \)

\[ \frac{a^2 + 9b^2}{a} = 9 \]

\[ a^2 = 39 \]

\[ a = \pm 3 \]

\[ (\pm 3, 0) \]

\[ 12 - 8a = 0 \]

\[ a = \frac{3}{2} \]

\[ 9b^2 = 9 - \left( \frac{3}{2} \right)^2 \]

\[ b^2 = \frac{3}{4} \]

\[ b = \pm \frac{\sqrt{3}}{2} \]
So: \((\pm 3, 0) \quad \text{or} \quad (\frac{3}{2}, \pm \frac{\sqrt{3}}{2})\)

So:

4 points with normal line passing through P.

Remark: \( a \cdot b = 0 \quad (\text{only if } b \neq 0) \)

\( L \rightarrow a = 0 \)
Rational Exponents in the Power Rule

Example: \( f(x) = \sqrt{x} \quad f'(x) = \frac{1}{2 \sqrt{x}} \)

\[ f(x) = x^{\frac{1}{2}} \quad f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \]

Looks like integer power rule

What about \( 3\sqrt{x} \), \( x^{\frac{11}{7}} \) ... ?

We want: \( \frac{dy}{dx} \)

\( y = x^{\frac{p}{q}} \) \( p \), \( q \) integers
We want to change $y = x^{\frac{p}{q}} \ (i)$ into something with only integer exponents.

What do we do?

$$y = x^{\frac{1}{3}} \ (= 3\sqrt[3]{x})$$

$$(\cdot)^3 \ \rightarrow \ x^3 = x$$

Or in general:

$$y = x^{\frac{p}{q}}$$

$$(\cdot)^q \ \rightarrow \ y^q = x^p$$
\[
\frac{d}{dx} (y^q) = \frac{d}{dx} (x^p) \\
q \cdot y^{q-1} \frac{dy}{dx} = p \cdot x^{p-1} \\
(\text{usual integer power rule})
\]

\[
\frac{dy}{dx} = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}}
\]

\[
y = x^{\frac{p}{q}}
\]

Plug in: \[
\frac{dy}{dx} = \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{\frac{p}{q}})^{q-1}}
\]

\[
= \frac{p}{q} \cdot x^{p-1} - \left(\frac{p}{q}\right)(q-1)
\]

\[
p-1 - \left(\frac{p}{q}\right)(q-1) = p-1 - p + \frac{p}{q}
\]

\[
= \frac{p}{q} - 1
\]
We have \( \frac{dy}{dx} = \frac{P}{q} x^{\frac{P}{q} - 1} \).

\[
\left( x^{\frac{P}{q}} \right)' = \frac{P}{q} x^{\frac{P}{q} - 1}
\]

\( r = \frac{P}{q}, \quad r \text{ rational} \)

\[
\left( x^r \right)' = r x^{r-1}
\]

Power rule for rational exponents.

Ex.: \( f(x) = \sqrt[3]{x^2 + 1} = (x^2 + 1)^{\frac{1}{3}} \)

\( f'(x) = \frac{1}{3} (x^2 + 1)^{-\frac{2}{3}} \cdot 2x \)
Ex: \((x^{-\frac{5}{27}})^1 = -\frac{5}{27} \times x^{-\frac{5}{27}} - 1\)

Differentiating \(\ln(x)\)

\[ y = \ln(x) \]

Search for \(\frac{dy}{dx}\).

\[ e^{\ln(x)} = x \]

\[ \frac{d}{dx} (e^y) = \frac{d}{dx} (x) \]

\[ e^y \cdot \frac{dy}{dx} = 1 \]

\[ \frac{dy}{dx} = e^{-y} \]
\[ y = \ln(x) \]

\[ e^{-y} = e^{-\ln(x)} = \frac{1}{e^{\ln(x)}} = \frac{1}{x} \]

So:

\[ \frac{dy}{dx} = \frac{1}{x} = (\ln(x))' \]

Derivative of \( \ln(x) \)

Ex: \( f(x) = \ln(4x) \)

\[ f'(x) = \frac{1}{4x} \cdot (4x)' \]

\[ = \frac{1}{4x} \cdot 4 = \frac{1}{x} \]
Note: \( \ln(4 \cdot x) = \ln(4) + \ln(x) \)

So: \((\ln(4-x))' = (\ln(4))' + (\ln(x))' \)
\[ = \frac{1}{x} \]

In general: \((\ln(c \cdot x))' = \frac{1}{x} \)