Ex: Making a piecewise function
dill

\[ f(x) = \begin{cases} 
  a \cdot x^2 + 1 & x < 1 \\
  2 & x = 1 \\
  b \cdot x^3 + c & x > 1 
\end{cases} \]

At \( x = 1 \): need this to be continuous:

glim_{x \to 1-} f(x) = g(x) = f(1)
glim_{x \to 1+} f(x) = f(1)

\[ a = 1 \]
\[ b + c = 2 \]
How to avoid this:

Graph of $f'$:

\[ \lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x) \]
\[
\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} \frac{d}{dx} (ax^2 + 1) = 2ax
\]

\[
= 2a
\]

\[
\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} \frac{d}{dx} (bx^3 + c) = 3bx^2 + c
\]

\[
= 3b + c
\]

So:

\[
2a = 3b + c
\]

\[
2a = 2b + b + c
\]
\[ a = 1 \quad b + c = 2 \]

\[ Z = 26 + (b + c)^2 = 2b + 2 \]

\[ 2b = 0 \]

\[ \Rightarrow b = 0 \]

\[ \Rightarrow c = 2 \quad (b + c = 2) \]

Summary:
1) Make \( f \) cts
2) Make \( f' \) cts

(Evaluate one-sided limits, make them equal)
Midterm

16th Oct (7pm - 8pm)

Content:
- Check the common webpage
- We have now covered all of it

Closes: will receive an email
The Chain Rule

Ex: \[ f'(x) = \sqrt{x^2 + 1} \]

\[ f_1(x) = (x + 1)^{100} \]

\[ f_2(x) = \sqrt{x} \]

\[ f_3(x) = e^x \]

\[ g(x) = \sin(\cos(x)) \]

All of these are compositions:

\[ f(x) = g(f(h(x)) \]
Ex: \((x + 1)^{100}\)
\[
X \rightarrow x + 1 \rightarrow (x + 1)^{100}
\]

\[
h(x) = \frac{x + 1}{x^{100}}
\]

\[
g(x) = \frac{x^{100}}{x}
\]

\[
f(x) = (x + 1)^{100} = g(h(x))
\]

Ex: \(\sqrt{x^2 + 1}\)
\[
x \rightarrow x^2 + 1 \rightarrow \sqrt{x^2 + 1}
\]

\[
g(x) = \frac{x^2 + 1}{x}
\]

\[
h(x) = \sqrt{x}
\]

\[
\sqrt{x^2 + 1} = g(h(x))
\]

\[
h(g(x))
\]
What we need:

How to differentiate

\[ f(x) = g(h(x)) \]

Chain Rule:

\[ f'(x) = g'(h(x)) \cdot h'(x) \]

When \( h \) is differentiable at \( x \),
\( g \) is differentiable at \( h(x) \).

**Example:** \( f(x) = (x+1)^{100} = g(h(x)) \)

\[ g(x) = x^{100} \quad \text{and} \quad g'(x) = (100x^{99}) \]
\[ h(x) = x + 1 \quad \text{and} \quad h'(x) = 1 \]

\[ f'(x) = g'(h(x)) \cdot h'(x) = 100(x+1)^{99} - 100(x+1)^{99} \]
Ex: \[ f(x) = h(x)^n \]

[Note: if \( h(x) = x \), this is the power rule \( (x^n)' \).]

So we need \( g(x) \) with

\[ f(x) = g(h(x)) \]

\[ g(x) = x^n \quad g'(x) = nx^{n-1} \]

So:

\[ f'(x) = g'(h(x)) \cdot h'(x) \]

\[ (h(x)^n)' = nh(x)^{n-1} \cdot h'(x) \]
"Generalized Power Rule"

Ex: \[ f(x) = \sin(x)^{100} \]

\[ f'(x) = 100 \sin(x)^{99} \cdot (\sin(x))' \]

\[ = 100 \cos(x) \sin(x)^{99} \]

Ex: \[ f(x) = e^{\sin(x)} \]

\[ g(x) = e^x \quad \uparrow \quad h(x) = \sin(x) \]

\[ \text{"inner function"} \quad \uparrow \quad \text{"outer function"} \]
\[ f'(x) = g'(h(x)) \cdot h'(x) \]

\[
= \sin(x) \\
= e^{\sin(x)} \cdot \cos(x)
\]

**Ex:** \[ f(x) = \sin(\cos(x)) \]

\[ f'(x) = \cos(\cos(x)) \cdot (-\sin(x)) \\
= -\cos(\cos(x)) \cdot \sin(x) \]

**Q:** \[ \cos^2(x) \quad vs \quad \cos(\cos(x)) \]

\[ = (\cos(x))^2 \quad \checkmark \\
= \cos(x) \cdot \cos(x) \]

Same with \[ \sin(\sin(x)) \neq \sin^2(x) \]
Alternative Notation

\[ y = f(u), \quad u = g(x) \]

i.e. \[ y = f(g(x)) \]

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

\[ f'(g(x)) \cdot g'(x) \]

To remember it: EXPanding the fraction
Ex: \[ y = (3x + 2)^n \]

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

Let \( u = 3x + 2 \)

Then \( y = u^n \)

So \( \frac{dy}{dx} = 11u^{10} \cdot 3 \)

\( = 33u^{10} \)

\( = 33(3x+2)^{10} \)
Example:

a) \( y = \cos^3 x = (\cos x)^3 \)

b) \( y = \cos(x^3) \)

Differentiate:

a) \[
\frac{dy}{dx} = 3(\cos x)^2 \cdot \frac{d}{dx}(\cos x)
\]

\[= 3(\cos x)^2 \cdot (-\sin x)\]

b) \[
\frac{dy}{dx} = -\sin(x^3) \cdot \frac{d}{dx}(x^3)
\]

\[= -\sin(x^3) \cdot 3x^2\]

\( \neq a) \)
Three or more functions

Ex: \[ f(x) = e^{\sin(x^2)} \]

\[ = g(h(x)) \]

\[ h(x) = \sin(x^2) \]
\[ g(x) = e^x \]

\[ f'(x) = g'(h(x)) \cdot h'(x) \]
\[ = e^{\sin(x^2)} \cdot (\sin(x^2))' \]

\[ \sin(x^2) = h_1(h_2(x)) \]
\[ h_1(x) = \sin(x) \]
\[ h_2(x) = x^2 \]
\[ e^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x = e^{\sin(x^2)} \cdot f(g(h(x))) \]

\[ f(x) = e^x, \quad g(x) = \sin(x), \quad h(x) = x^2 \]

\[ (f'(g(h(x)))) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) \]

(Chain rule for three functions)
In fraction notation

\[ Y = \frac{\partial}{\partial u_1} \]

\[ u_1 = u_1(u_2) \]

\[ u_2 = u_2(u_3) \]

\[ \vdots \]

\[ u_{10} = u_{10}(u_{11}) \]

\[ \frac{dy}{dx} = \frac{dy}{du_1} \cdot \frac{du_1}{du_2} \cdot \ldots \cdot \frac{du_{10}}{dx} \]

with 11

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

with 2
Ex: \( f(x) = \cos(\cos(\cos(\cos(x)))) \)

\[
f'(x) = -\sin(\cos(\cos(\cos(x)))) \\
\cdot (-\sin(\cos(\cos(x)))) \\
\cdot (-\sin(\cos(x))) \\
\cdot (-\sin(x)) \\
= \sin(\cos(\cos(\cos(\cos(x)))) \cdot \sin(\cos(\cos(x)))) \\
\cdot \sin(\cos(x)) \cdot \sin(x)\]