Lecture 1

(Chapter 1.3 in text)

Exponential functions

Start with $1$, double every year

<table>
<thead>
<tr>
<th>Year</th>
<th>Money</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>$8$</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
</tbody>
</table>
How much money do we have after \( n \) years?

\[ a \rightarrow a^n \] amount of money

This is an exponential function.

Def: \( f(x) = b^x \), \( b > 0 \)

is called an exponential function.
Graph:

\[ f(x) \]

\[ f(x) = 2 \times x \]

Example:

\[ g(x) = \left(\frac{1}{2}\right)^x \]
\[ f(x) = \left( \frac{1}{2} \right)^x \]

<table>
<thead>
<tr>
<th>[ x ]</th>
<th>[ \frac{1}{2} ]</th>
<th>[ 1 ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{1}{4} ]</td>
<td>[-\frac{1}{2} ]</td>
<td>[ 2 ]</td>
</tr>
<tr>
<td>[ \frac{1}{8} ]</td>
<td>[-\frac{3}{4} ]</td>
<td>[ 3 ]</td>
</tr>
<tr>
<td>[ \frac{1}{16} ]</td>
<td>[ -1 ]</td>
<td>[ 4 ]</td>
</tr>
</tbody>
</table>

Properties:

1) \( f(x) \) is defined for all real \( x \).

2) \( f(0) = 1 \)

3) Increasing if \( b > 1 \), decreasing if \( b < 1 \)
Rules:
- \( b \times b^y = b^{x+y} \)
- \( b^0 = 1 \)
- \( \frac{b^x}{b^y} = b^{x-y} \)
- \( (b^x)^y = b^{x \cdot y} \)

Example:
- \( 2^2 \cdot 2^3 = 2^5 \)

\((4 \cdot 8 = 32)\)

- \( 3^2 \cdot 4 = 9 \cdot 4 = 36 \)

- \( (3^2)^4 = 3^8 \)

- \( (2^{-1})^3 = 2^{-3} \)
- \( (\frac{1}{2})^3 = \frac{1}{8} \)
Natural Exponential

\[ f(x) = e^x \]
\[ e \approx 2.718 \ldots \]
Euler's number

This appears everywhere in calculus.

Inverse Functions

Ex: Say we doubled $1 \times x$ times and got $\$512$, how often did we double?

i.e. \[ 2^x = 512 \]
Or: \( f(x) = 2^x \)

Then we want \( f(x) = 512 \)

If we have a function \( f^{-1}(x) \) ("inverse function")

such that \( f^{-1}(f(x)) = x \).

So: \( f^{-1}(f(x)) = x = f^{-1}(512) \),

(Aside: \( f^{-1}(x) = \log_2 x \))
One-to-one functions

\( f(x) \): one \( x \) such that
\[ f(x) = 1 \]

\( g(x) \): three different \( x \) such that
\[ g(x) = 1 \]
Here:

- \( f(x) \) is one-to-one
- \( g(x) \) is not one-to-one

**Def:** A function is called one-to-one if every horizontal line intersects its graph at most once.
Inverse functions and one-to-one:

Fact: Let \( f(x) \) be one-to-one, then it has a (unique) inverse function \( f^{-1}(x) \) satisfying \( f^{-1}(f(x)) = x \).

Ex: \[ x^2 = 9 \]
\[ x = \pm 3 \]
\[ = \pm \sqrt{9} \]
If we restrict to $x \geq 0$:

\[ f(x) = x^2, \quad x \geq 0 \]

\[ f^{-1}(x) = \sqrt{x} \]

**Def:** $f(x)$ is one-to-one on an interval $[a, b]$ if its graph restricted to $[a, b]$ is one-to-one.

$[a, b]$: 

\[ a \quad \text{---------} \quad b \]
Ex: \( f(x) = x^2 \)

- on \( \mathbb{C} \cup (\infty) \):
  \( x^2 \) is one-to-one

- on \( (-\infty, 0] \):
  \( x^2 \) is one-to-one

Ex: \( f(x) = \sin(x) \)

This is one-to-one on \( [-\frac{\pi}{2}, \frac{\pi}{2}] \).