Lecture 9

Recall:

Max/min of a function
\[ f(x,y) \]
subject to \[ g(x,y) = 0 \].

Then: solve

\[ \begin{align*}
  g(x,y) &= 0 \\
  \nabla f(x,y) &= \lambda \nabla g(x,y)
\end{align*} \]

Ex: \[ f(x,y) = 2x^2 + y^2 + 2 \]

\[ g(x,y) = x^2 + 4y^2 - 4 \]

\[ g = 0: \]

\[ \begin{array}{c}
  -2 \\
  -1 \\
  1 \\
  2
\end{array} \]
\[ \nabla g = \langle 2x, 8y \rangle \]
\[ \nabla f = \langle 4x, 2y \rangle \]

get:

\[ \begin{cases} x^2 + 4y^2 - 4 = 0 & \text{(1)} \\ 24x = 2x \lambda & \text{(2)} \\ 2y = 8y \lambda & \text{(3)} \end{cases} \]

(2): \[ 0 = x(2\lambda - 4) \]

Case 1: \[ x = 0 \]
Case 2: \[ \lambda = 2 \]

For 1: \[ x = 0 \]  \[ \begin{cases} 4y^2 = 4 & \text{(1)} \\ y = \pm 1 \end{cases} \]

(3): \[ z = 8x \]  \[ \lambda = \frac{1}{4} \]
get 2 solutions from case 1:

\[ x = 0, \ y = \pm 1, \ x = \frac{1}{4} \]

Case 2: \( x = 2 \)

(3): \( 2y = 8 \cdot 2 \)

\[ 2y = 16y \]

\[ \Rightarrow y = 0 \]

(1): \( x^2 = 4, \ x = \pm 2 \)

2 more solutions:

\[ x = \pm 2, \ y = 0, \ y = 2 \]
Now compute $f$ at each of those:

$f(0, \pm 1) = 2 \cdot 0 + 1 + 2 = 3$

$f(\pm 2, 0) = 2 \cdot (2)^2 + 0 + 2 = 10$

Conclusion: $3$ is the min of $f$, $10$ is the max of $f$.

Subject to $g(x_1, y) = 0$. 
Note: Lagrange multiplier method will find associate max (min of $f$ on $g(x,y) = 0$.

Why should this work? (Not examine)

Level curves of $f$ should be parallel to $g(x,y) = 0$ at max or min.
So then: \( \nabla f \perp \text{level curve of } f \) 
\( \nabla g \perp g(x,y) = 0 \)

Now: \( \nabla f \parallel \nabla g \) is the same as level curve of \( f \) being parallel to \( g(x,y) = 0 \) at some point.

\[ \int \nabla f \cdot (x,y) = \iint \nabla g(x,y) \]
\[ \Rightarrow g(x,y) = 0 \]
Ex (2012 Final)

The demand curve for some quantity satisfies

\[ p^2 + 4q^2 = 800 \]

where \( p \) is price, \( q \) quantity sold (per day).

What is the maximum revenue?

Use Lagrange multipliers!

price: price per unit of some good

quantity: number of goods sold

revenue: \( R \) total amount taken in
\[ R = p \cdot q \]

Solution:
\[ f(p, q) = p \cdot q \]
\[ g(p, q) = p^2 + 4q^2 - 800 \]

Lagrangian Multipliers:

\[ \begin{cases} p^2 + 4q^2 - 800 = 0 \\ \langle q, p \rangle = \lambda < q, p > \end{cases} \]
\[ \begin{cases} \lambda q = 2p \\ \lambda p = 8q \end{cases} \]

We can try \( \frac{(2)}{(3)} \):
\[
\frac{q}{p} = \frac{x}{z} \frac{2q}{8q}
\]

Can \( p = 0 \), \( x = 0 \), \( q = 0 \)?

\( p = 0 \): \((2)\): \( q = \frac{y}{2p} \)
\( q = 0 \)

But then \((1)\) cannot be true,

\( x = 0 \): not possible

\( q = 0 \): not possible

So:
\[
\frac{q}{p} = \frac{2q}{8q}
\]

\[
8q^2 = 2p^2
\]

\[
p^2 = 4q^2
\]
\[ 4q^2 + 4q^2 = 800 \]
\[ q^2 = 100 \]
\[ q = \pm 10 \]
\[ q = 10 \] as \( q > 0 \)

(should produce positive number of goods)

\[ p^2 = 800 - 4q^2 \]
\[ = 400 \]

So: \( p = \pm 20 \)
\[ p = 20 \] (as \( p > 0 \))

Can check \( x \):
\[ x = \frac{q}{2p} = \frac{10}{40} = \frac{1}{4} \]
So: \( p = 20, q = 10 \) is our solution,

\[ R = 1 \cdot q = 20 \cdot 10 = 200 \]

Utility functions

\( l \): leisure time
\( g \): amount of consumable goods

\[ U = U(l, g) \]

IDEA: Want to maximize \( U \)
subject to a constraint on \( l, g \).
1) $U$ increases if $l$ or $g$ increase.

2) $l$, $g$ are inversely related:

- $l \uparrow$, $g \downarrow$
- $l \downarrow$, $g \uparrow$

Ex: $U = f(l, g) = e^{\frac{l}{3}} g^{\frac{2}{3}}$

Subject to a constraint

$h(l, g) = 3l + 2g - 12 = 0$

What should $l, g$ be to maximize $U$?
\[ 3 \ell + 2 g - 12 = 0 \]
\[ \nabla f = \lambda \nabla h \]
\[ \nabla f = \begin{pmatrix} -\frac{2}{3} \ell \frac{2}{3} \frac{2}{3} e^{\frac{1}{3}} g^{\frac{1}{3}} \end{pmatrix} \]
\[ \nabla h = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix} \]
\[ \begin{align*}
3 \ell &+ 2 g - 12 = 0 \\
\frac{1}{3} e^{-\frac{2}{3}} g^{\frac{2}{3}} &= \lambda \cdot 3 \\
\frac{2}{3} e^{\frac{1}{3}} g^{\frac{1}{3}} &= \lambda \cdot 2
\end{align*} \]
IDEA: again do \( \frac{2}{3} \)

we won't divide by 0.
as \( \delta \to 0 \)

\[
\left( u = e^{\frac{1}{3}} q^{\frac{2}{3}} = 0 \right)
\]

if \( \delta = 0 \) or \( q = 0 \)

Finish Wednesday