Ex: We have a factory with production input $P$. Assume $P = P(C, L)$.

$C$: Capital investment
$L$: Money spent on Labour

Q: What do $\frac{\partial P}{\partial C}$ and $\frac{\partial P}{\partial L}$ represent?
Ex: \( f(x, y) = \frac{x}{x+y} \)

\[ \Delta y = -\frac{x}{(x+y)^2} \]

(Chain rule or Quotient rule)

\[ f_x = \frac{1 \cdot (x+y) - 1 \cdot x}{(x+y)^2} \]

\[ = \frac{y}{(x+y)^2} \]

Higher Order partial derivatives

One variable: \( f(x), f'(x), f''(x), f'''(x), \ldots \)
Two variables:

\[
\frac{\partial}{\partial x} f(x, y) = f_x(x, y)
\]

\[
\frac{\partial}{\partial y} g(x, y) = g_y(x, y)
\]

First order partials:

\[
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} f(x, y) \right) = f_{xx}(x, y)
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} g(x, y) \right) = g_{yy}(x, y)
\]

Second order partials:

\[
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} g(x, y) \right) = g_{yx}(x, y)
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} f(x, y) \right) = f_{xy}(x, y)
\]

\[
\frac{\partial^2}{\partial x^2} f(x, y) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} f(x, y) \right)
\]

\[
\frac{\partial^2}{\partial y^2} g(x, y) = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} g(x, y) \right)
\]

\[
\frac{\partial^2}{\partial y \partial x} g(x, y) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} g(x, y) \right)
\]

\[
\frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} f(x, y) \right)
\]
Ex: $f(x,y) = 3x^4y - 2xy + 5xy^3$

Calculate all second order partials:

- $\frac{\partial x}{\partial x} = 12x^3y - 2y + 5y^3$
- $\frac{\partial y}{\partial x} = 3x^4 - 2x + 15xy^2$

$\frac{\partial x}{\partial x} = 36x^2y$  

$\frac{\partial y}{\partial y} = 12x^3 - 2 + 15y^2$

$\frac{\partial y}{\partial x} = 30xy$

$\frac{\partial y}{\partial x} = 12x^3 - 2 + 15y^2$

\[
\frac{\partial}{\partial x} \left( y + \sin(y) + e^y \right) = 0
\]
Observation: $f_{xy} = f_{yx}$ in this example.

FACT (Cairns' Theorem)

If $f(x)$ is a twice differentiable function and $f_{xy} = f_{yx}$ are continuous, then $f_{xy} = f_{yx}$.

Think of it as: if $f$ is “nice” then $f_{xy} = f_{yx}$. 
Ex: Can there be a twice differentiable function \( f(x, y) \) ("nice") with

\[
x_x = 3y \quad , \quad x_y = 2x^2
\]

\[
y_x y = 3 \quad , \quad y_y x = 2
\]

So this is not possible, as

\[
x_x + f_y x
\]
Local extrema

Recall:

\( f' = 0 \) at local max/min
Def. A local max at \((a,b)\)
for the function \(f(x,y)\) means
\[ f(x,y) \leq f(a,b) \text{ for } (x,y) \text{ close to } (a,b). \]

A local min:
\[ f(x,y) \geq f(a,b) \text{ for } (x,y) \text{ close to } (a,b). \]
What about \( f(x, y) \) at \((a, b)\)?

Fix \( y = b \) (local max)

So: \((a, b)\) is still a local max

when we allow \( x \) to vary, but

fix \( y \).

So: \( \Delta_x \left( a, b \right) = 0 \)

Same reasoning:

\( \Delta_y \left( a, b \right) = 0 \)

Therefore: \((a, b)\) local max/min

\( \Rightarrow \Delta_x \left( a, b \right) = \Delta_y \left( a, b \right) = 0 \)
Ex: \( f(x, y) = x^2 + y^2 \)

So compute \( \nabla f \):

\[
\begin{align*}
\nabla_x f &= 2x, \\
\nabla_y f &= 2y
\end{align*}
\]

So: \( \nabla_x f = 0, \nabla_y f = 0 \)

We see \( x = y = 0 \).

So this is consistent with the picture.
Ex: \( A(x, y) = x^2 - y^2 \)

\[
\begin{align*}
&\text{(set } y = 0: \quad A(x, y) = x^2 \\
&x = 0: \quad A(x, y) = y^2) \\
\end{align*}
\]

\( \hat{b}_x = 2x, \quad \hat{b}_y = -2y \)

So at \((0, 0)\): \( \hat{b}_x = \hat{b}_y = 0 \)

Note: \((0, 0)\) is not local max/min

It is a saddle point.
Other things that prevent local max/min:

\[ f(x) = x^3 \]

\[ f'(x) = 3x^2 \]

\[ f''(x) = 6x, \quad f''(0) = 0 \]

So second derivative test is inconclusive.
\[ f(x, y) = x^3 \]

\[ g(x, y) = x^3 \]

Here: \( f_y = 0 \) \( \cdot \) \( f_x = 3x^2 \)

At \( x = 0 \): \( f_x = f_y = 0 \)

But not local max/min.