The goal is to interpret automorphic forms as algebraic sections of algebraic vector bundles on Shimura varieties $\Gamma \backslash G(\mathbb{R})/K$.

We focus on the case $G = \text{Sp}_{2g}$.

**Motivation:** Use tools from algebraic geometry (sheaves+cohomology) to study automorphic forms (dimension formulas via Riemann-Roch).

Let $f$ be a classical weight $K$ modular form $\Gamma \subset \text{SL}_2(\mathbb{Z})$. Define $F : \text{SL}_2(\mathbb{R}) \to \mathbb{C}$ by $F(g) = j(g, i)^{-K} f(g \cdot i)$.

We check how $F$ behaves under a representation of $\text{SO}(2)$ on the right. We get $F(\gamma g \cdot K \theta) = e^{iK\theta} F(g)$ for all $\gamma \in \Gamma, g \in \text{SL}_2(\mathbb{R})$ and $K \theta \in \text{SO}(2)$.

Let $G$ be a Lie group, $K \subset G$ a closed Lie subgroup (assume $K$ connected). The map $G \to G/K$ has the structure of a principal $K$-bundle i.e.

1. Free right $K$-action on $G$.
2. Fibers are diffeomorphisms to $K$.
3. Locally trivial.

Let $(V, \sigma)$ be an $r$-dimensional complex representation of $K$. Define $E_K := G \times_K V = G \times V/K$. The right action of $K$ on $G \times V$ is given by $k \cdot (g, v) \mapsto (gk, \sigma(k^{-1})v)$.

**Fact:** $\Pi : E_V \to G/K$ is a rank $r$ smooth (complex) vector bundle (it is an associated vector bundle).

**Definition.** We say $\Pi : E \to X$ is a smooth complex rank $r$ vector bundle if $\Pi$ is a smooth surjection between smooth manifolds such that

1. $\Pi^{-1}(x) \cong \mathbb{C}^r$
2. There is an open cover $\{U_i\}$ of $X$ with diffeomorphisms $\phi_i : \Pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ with compatibility condition $g_{i,j} =
\(\phi_i \circ \phi_j^{-1} : U_i \cap U_j \to GL_r(\mathbb{C})\), this map is called a transition function. Those maps must satisfy **Cocycle condition.** \(g_{ij} = g_{ik}g_{kj}\) on \(U_i \cap U_j \cap U_k\). Also we need the composition of \(\phi_i\) with projection on the first coordinate to be equal to \(\Pi\) on \(P_i^{-1}(U_i)\).

For associated bundles, the triangle commutes

\[
\begin{array}{c}
\text{GL}(V) \\ \searrow \\
K \leftarrow U_i \cap U_j \rightarrow g_{ij} \rightarrow \text{GL}(V).
\end{array}
\]

**Remarks.** If \(X\) is a complex manifold or algebraic variety, a holomorphic (resp. algebraic bundle) is such that the transition functions \(g_{ij}\) are holomorphic (resp. algebraic) + likewise for \(\Pi\).

**Definition.** A smooth section of a vector bundle \(\Pi : E \to X\) is a smooth map \(s : X \to E\) such that \(\Pi \circ s = \text{Id}_X\).

A holomorphic (resp. algebraic) section of a holomorphic (resp. algebraic) bundle is a section where \(s\) is also holomorphic (resp. algebraic). The set of smooth sections of a vector bundles is usually denoted \(\Gamma_{C^\infty}(E)\).

**Key Lemma.** We have the bijection

\[
\{ F : G \to V | F(g) = \sigma(k^{-1})F(g) \ \forall g \in G, \ \forall k \in K \} \xrightarrow{\cong} \Gamma_{C^\infty}(E_v),
\]

\(F \mapsto s\), where \(s(gK) = [(g, F(g))]\)
conversely \(s \mapsto F\), defined by \(F(g) = v\), where \(s(gK) = [(g, v)]\).

**Main example.** Let \(G\) be a connected semisimple algebraic group and \(K \subset G(\mathbb{R})\) denote a maximal compact subgroup.

Suppose that \(X = G(\mathbb{R})/K\) is a Hermitian symmetric domain.

Let \((V, \sigma)\) be a representation of \(K\). Let \(E_v = G(\mathbb{R}) \times_K V \to X\) the associated smooth complex vector bundle over \(X\). Let \(\Gamma \subset G(\mathbb{R})\) a torsion-free lattice. Define a \(\Gamma\)-action on \(G(\mathbb{R}) \times V\) by \(\gamma \cdot (g, v) \mapsto (\gamma g, v)\).

Let \(E_{V,\Gamma} = \Gamma \backslash G(\mathbb{R}) \times_K V \to \Gamma \backslash X\) is a smooth complex vector bundle on \(\Gamma \backslash X\).

\[
\Gamma_{C^\infty}(E_{V,\Gamma}) = \{ F : G(\mathbb{R}) \to V | F(\gamma \cdot g \cdot k) = \sigma(k^{-1})F(g) \ \forall \gamma \in \Gamma, \ g \in G(\mathbb{R}), \ k \in K \}.
\]

These “formally” look like what we would call vector-valued automorphic forms, except \(E_{V,\Gamma}\) and its section are apriori smooth, not algebraic.
Borel Embedding.

A Hermitian symmetric domain $X = G(\mathbb{R})/K$ embeds into its “compact dual” $\hat{X} = G(\mathbb{C})/P$. The map $\beta X \to \hat{X}$ is the Borel embedding.

Here $P$ is some parabolic subgroup of $G(\mathbb{C})$ (i.e. $G(\mathbb{C})/P$ is a smooth projective variety, “generalize flag variety”).

Example. Take $G = SL_2$, we have $H = SL_2(\mathbb{R})/SO(2)$.

Let $K = SO(2) = Stab_{SL_2(\mathbb{R})}(i)$. In $SL_2(\mathbb{C})$, we have that
\[
\begin{pmatrix} 0 & -1 \\ 1 & -i \end{pmatrix} \in SL_2(\mathbb{C})
\]
takes $i$ to $\infty$.

\[
\text{Stab}_{SL_2(\mathbb{C})}(\infty) = P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}.
\]

Flags. A flag is a sequence of vector spaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_s = \mathbb{C}^n$.

Let $n_i = \dim V_i/V_{i-1}$, $n = n_1 + \cdots + n_s$. The stabilizer of this flag is a block triangular matrix with $s$ diagonal blocks (in the appropriate basis), where the $i$th is of size $n_i$. This is the parabolic subgroup stabilizing the flag.

Let $Y_1 = \{ \{O\} \subset L \subset \mathbb{C}^2 | \dim(L) = 1 \}$. Then $SL_2(\mathbb{C})$ acts transitively on $Y_1$.

The standard flag : $\{0\} \subset \text{span}_{\mathbb{C}}(e_1) \subset \mathbb{C}^2$. Its stabilizer is $P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$.

We have $Y_1 \cong SL_2(\mathbb{C})/P \cong \mathbb{P}^1(\mathbb{C})$, the compact dual of $\mathbb{H}$.

Example. $G = Sp_{2g}$, $X = \mathbb{H}_g = Sp_{2g}(\mathbb{R})/U(g)$

Consider $(\mathbb{Z}^{2g}, \langle \cdot, \cdot \rangle)$ the symplectic lattice with basis $\{e_1, \ldots, e_g, f_1, \ldots f_g\}$, where $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle$ and $\langle e_i, f_j \rangle = \delta_{ij}$.

We have $(\mathbb{Z}^{2g}, \langle \cdot, \cdot \rangle) \cong (\mathbb{Z}^{2g}, \langle \cdot, \cdot \rangle) \otimes \mathbb{C}$.

Let
\[
Y_g = \{ \{0\} \subset L \subset \mathbb{C}^{2g} | \dim(L) = g, \ (x, y) = 0 \ \forall x, y \in L \}.
\]

$L$ here is a Lagrangian, i.e. maximal totally isotropic subspace.

$Y_g$ is the Grassmannian of Lagrangian subspaces, and $Sp_{2g}(\mathbb{C})$ acts transitively on $Y_g$.

Standard flag. $0 \subset (e_1, \ldots, e_g) \subset \mathbb{C}^{2g}$. The stabilizer of the standard flag is
\[
\begin{bmatrix} A & * \\ 0 & (A^{-1})^T \end{bmatrix} \in Sp_{2g}
\]

Therefore, $Y_g \cong Sp_{2g}/P$.

We have an open set $Y_g^+ = \{ L \in Y_g | -i(x, \overline{x}) > 0 \ \text{for all} \ x \in L \}$. 

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$\text{Sp}_{2g}(\mathbb{R})$ acts transitively on $Y_g^{+}$,
\[
\langle x, \overline{x} \rangle_{g \in \text{Sp}_{2g}(\mathbb{C})} \equiv \langle gx, g\overline{x} \rangle_{g \in \text{Sp}_{2g}(\mathbb{R})} \equiv \langle gx, g\overline{x} \rangle.
\]

$-i\langle x, x \rangle = i\langle x, x \rangle$, it is a hermitian form. The stabilizer in $\text{Sp}_{2g}(\mathbb{R})$ of a point in $Y_g^{+}$ is $U(g)$.

**Proposition.** We have a 1–1 correspondence

\[
\{\text{Smooth bundles } G(\mathbb{R}) \times_K V \to X\} \leftrightarrow \{\text{Algebraic bundles } G(\mathbb{C}) \times_P V \to \hat{X}\}
\]

- **Left-to-right :** Extend $(V, \sigma)$ to $(V, \tilde{\sigma})$.
- **Right-to left :** Restrict via $\beta : X \to \hat{X}$.

**Proof.** Given a representation $(V, \sigma)$ of $K$, by the Weyl unitary trick we get a unique unitary representation of $K_{\mathbb{C}}$. We use the Levi decomposition $P = U \rtimes M$ where $M \cong K_{\mathbb{C}}$ (the particular levi factor here is the complexification of $K$). We can pullback the representation on $M$ to $P$ (equiv define a representation of $P$ where $U$ acts trivially). So we get $G(\mathbb{C}) \times_P V \to \hat{X}$ as an algebraic bundle.

**Corollary.** Pulling back via $\beta$, we realize $G(\mathbb{R}) \times_K V \to X = G(\mathbb{R})/K$ to be holomorphic.

It was immediate previously that since $G(\mathbb{R}) \times_K V \to X$ is a holomorphic bundle, then $E_{\nu, \Gamma} \to \Gamma \backslash X$ is one as well (we just quotient by $\Gamma$, which is discrete so it stays holomorphic).

**Bailey-Borel Theorem.** $\Gamma \backslash X$ is a quasi-projective algebraic variety (called a Shimura variety), and $E_{V, \Gamma}$ are algebraic bundles.