Talk 2 : Review of $\text{Sp}_{2n}(\mathbb{R})$, root systems and Cartan involution.

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0. Definition of $\text{Sp}_{2n}$ over $\mathbb{C}$ and $\mathbb{R}$.

Definition. The symplectic group

$$\text{Sp}_{2n}(\mathbb{K}) = \{ X \in \text{SL}_{2n}(\mathbb{K}) : X^T J X = J \},$$

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. A block matrix $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_{2n}(\mathbb{K})$ if and only if $A^T C = C^T A$, $B^T D = D^T B$, and $A^T D - C^T B = 1$.

Note that this is equivalent to the orthogonal group, seen as automorphism group of the standard quadratic form, where this is the automorphism group of the standard symplectic form on $\mathbb{K}^{2n}$ where $[(q,p), (q', p')] \mapsto p \cdot q' - q \cdot p'$, which has Gram matrix $J$.

We can also define the group of similitudes

$$\text{GSp}_{2n}(\mathbb{K}) = \{ X \in \text{SL}_{2n}(\mathbb{K}) : J X^T J X \in \mathbb{K} \},$$

so $X \in \text{GSp}_{2n}(\mathbb{K})$ if $X^T J X = t(X) J$ where $t(X) \in \mathbb{K}$. The map $X \mapsto t(X)$ is the multiplier, it is a character : $\text{GSp}_{2n} \to \text{GL}_1 = \mathbb{G}_m$.

We get the corresponding Lie algebra :

$$\mathfrak{sp}(\mathbb{K}) = \{ x \in \mathfrak{sl}_{2n}(\mathbb{K}) : X^T J + J X = 0 \}.$$

Note that the assumption $x \in \mathfrak{sl}_{2n}(\mathbb{K})$ is equivalent to taking $x \in \mathfrak{gl}_{2n}(\mathbb{K})$ when $\text{char}(\mathbb{K}) \neq 2$.

Recall that

$$\mathfrak{sl}_{2n}(\mathbb{K}) = \{ X \in \mathfrak{gl}_n(\mathbb{K}) : \text{Tr}(X) = 0 \}.$$

The unitary group is

$$U(n) = \{ X \in \text{GL}_n(\mathbb{C}) : U^* U = I_n \}.$$
where $U^* = U^T$ denotes the conjugate transpose. Again, the corresponding lie algebra is
\[ u(n) = \{ X \in \mathfrak{gl}_n(\mathbb{C}) : X^* + X = 0 \} . \]
The special orthogonal group is
\[ \text{SO}(n) = \{ X \in \text{GL}_n(\mathbb{R}) : X^TX = I_n \} , \]
and its Lie algebra
\[ \mathfrak{so}(n) = \{ X \in \mathfrak{gl}_n(\mathbb{R}) : X^T + X = 0 \} . \]
One can define $\text{Sp}(n) = \text{Sp}_{2n}(\mathbb{C}) \cap U(2n)$ the compact symplectic group, and $\mathfrak{sp}_{2n} = \mathfrak{sp}_{2n}(\mathbb{C}) \cap u(n)$.

We are interested in studying semisimple Lie algebras, i.e. algebras with no nontrivial bilateral ideal.

1. Root systems.

Let $G = \text{Sp}_{2n}(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$.

Let
\[ \mathfrak{h} = \{ H \in \mathfrak{g} : H = \text{diag}(h_1, \cdots, h_n, -h_1, \cdots, -h_n) \} , \]
it is an abelian subalgebra of $\mathfrak{g}$.

Let $E_{i,j}$ denote the canonical basis element of $\mathfrak{gl}_{2n}$. and $e_i \in \mathfrak{h}^*$ as $e_i(H) = h_i$.

For $H \in \mathfrak{h}$ we have a corresponding adjoint map $\text{ad}H : \mathfrak{g} \to \mathfrak{g} : X \mapsto [H, X]$ where $[,]$ denotes the usual Lie bracket $[A, B] = AB - BA$.

For $1 \leq i, j \leq n$ we have $\text{ad}H(E_{i,j}) = [H, E_{i,j}] = (e_j(H) - e_j(H))E_{i,j}$ so $E_{i,j}$ is a simultaneous eigenvector for all $\text{ad}H$. $H \in \mathfrak{h}$, with eigenvalues $(e_i - e_j)(H)$.

In general, if we let $1, j \leq 2n$ we get $\text{ad}H(E_{ij}) = (\pm e_i(H) \pm e_j(H))E_{ij}$.

Define the set of roots $\Delta = \{ e_i - e_j : 1 \leq i \neq j \leq n \} \cup \{ 2e_i : 1 \leq i \leq n \}$.

We have a root space decomposition
\[ \mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha , \]
where $\mathfrak{g} = \{ X \in \mathfrak{g} : (\text{ad}H)(X) = \alpha(X)X \ \forall H \in \mathfrak{h} \}$.

One has
\[
[g_\alpha, g_\beta] = \begin{cases}
g_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ 0 & \text{if } \alpha + \beta \text{ is not a root or } 0 \\ \subseteq \mathfrak{h} & \text{if } \alpha + \beta = 0
\end{cases}
\]

Figure 1: Root system for sl(2)

2. Cartan involution.

**Definition.** A linear connected *reductive* group is a closed connected subgroup of real/complex matrices that are stable under conjugate transpose. This is equivalent to the triviality of the unipotent radical (not trivial).

**Definition.** A linear connected *semisimple* group is a linear connected reductive group with finite centre.

**Examples.** $SL_n, SO_n, Sp_{2n}$ are all connected semisimple linear groups.

**Cartan involution.** Let $G$ be a linear connected semisimple group, define

\[
\Theta : G \to G \\
X \mapsto (X^*)^{-1}
\]

Note that we have $\Theta^2 = \text{id}$.

The differential of $\Theta$ at the identity is denoted by $\theta$ and it gives an automorphism of $\mathfrak{g}$, given by $\theta(X) = -X^*$. This $\theta$ is the *Cartan involution*, and is indeed an involution since $\theta(YX) = \theta(Y)\theta(X)$, $\theta(\lambda X) = \lambda \theta(X)$ and $\theta^2 = \text{id}$.

Clearly the polynomial $t^2 - 1$ kills $\theta$ so the eigenvalues of $\theta$ are $\pm 1$, let $\mathfrak{t}, \mathfrak{p}$ denote the eigenspaces of $1$ and $-1$ respectively. We have the descriptions :

$\mathfrak{t} = \text{Skew-Hermitian matrices and } \mathfrak{p} = \text{Hermitian matrices.}$

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This gives us the Cartan decomposition
\[ g = \mathfrak{k} \oplus \mathfrak{p}. \]

We have \([\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \text{ and } [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}\). In particular, \(\mathfrak{k}\) is a Lie subalgebra of \(g\).

**Proposition 1.** If \(G\) is a linear connected semisimple group, then \(g\) is semisimple.
If \(G\) is a linear connected reductive group then
\[ g = Z(g) \oplus [g, g], \]
where \(Z(g)\) denotes the centre of \(g\).

**Proposition 2.** If \(G\) is a real linear connected reductive group then \(K\) is compact and connected and it is a maximal compact subgroup of \(G\). Its Lie algebra is \(\mathfrak{k}\). The map
\[ K \times \mathfrak{p} \to G, \quad (k, x) \mapsto k \exp(X) \]
is a diffeomorphism onto \(G\).

Idea of the proof of connectedness of \(\text{SL}_n\): It acts transitively on the column vectors of \(\mathbb{C}^n \setminus \{0\}\). The subgroup fixing the last standard basis vector is
\[ \begin{bmatrix} \text{SL}_{n-1}(\mathbb{C}) & 0 \\ \mathbb{C}^{n-1} & 1 \end{bmatrix} \cong \text{SL}_{n-1}(\mathbb{C}) \rtimes \mathbb{C}^{n-1}. \]
We have
\[ \text{SL}_n(\mathbb{C})/(\text{SL}_{n-1}(\mathbb{C}) \rtimes \mathbb{C}^{n-1}) \cong \mathbb{C}^n \setminus \{0\}. \]
Then use induction.

### 3. Examples.

We can consider two involutions on \(\mathfrak{s\ell}_{2n}(\mathbb{C})\), the fixed points under the involution \(X \mapsto X\) is simply \(\mathfrak{s\ell}_{2n}(\mathbb{R})\).

The fixed vectors under the Cartan involution \(\theta\) are just \(\mathfrak{k} = \mathfrak{so}_2(\mathbb{R})\).

Both \(\mathfrak{sl}_2(\mathbb{R})\) and \(\mathfrak{su}(2) \cong \mathfrak{so}_3(\mathbb{R})\) “complexify” to \(\mathfrak{sl}_2(\mathbb{C})\), we say that the former is **split**.

In \(\mathfrak{sp}_{2n}(\mathbb{R})\), the maximum compact subalgebra is isomorphic to \(\mathfrak{u}(n)\).

In \(\mathfrak{sp}_{2n}(\mathbb{R})\), take \(X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\), the condition
\[ \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]
becomes
\[
\begin{bmatrix}
-C^T & A^T \\
D^T & B^T
\end{bmatrix} = \begin{bmatrix}
-C & -D \\
A & B
\end{bmatrix},
\]
so we get \(C = C^T, \ B = B^T\) and \(A^T = -D\).

And we have
\[
\begin{bmatrix}
A & B \\
C & -A^T
\end{bmatrix} = \begin{bmatrix}
-A^T & -C^T \\
B^T & A
\end{bmatrix},
\]
so the matrix is of the form \(\begin{bmatrix}
A & B \\
-C & D
\end{bmatrix}\), \(A, B\) being real matrices, which by considering the embedding \(\mathbb{C}\) into \(\mathfrak{gl}_2(\mathbb{R})\), we identify \(\begin{bmatrix}
A & B \\
-C & D
\end{bmatrix}\) which identifies our subalgebra of fixed points with \(\mathfrak{u}(n)\).

5. Follow up.

Let \(G \to \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\) defined by \((X, Y) = \text{Tr}(\text{ad}_X \text{ad}_Y)\).

Restriction to \(\mathfrak{k}, \mathfrak{p}\) is definite (so \(K \subset O(\mathfrak{j})\) or \(k\)).

Restriction to \(\mathfrak{p}\) is \(K\)-invariant so \(\mathfrak{p}\) can be identified with the tangent space \(T_e(G/K)\) with \(K\)-invariant inner product. So \(G/K\) has a \(G\)-invariant metric. \(G = K \exp(\mathfrak{p})\) so \(G/K \cong \exp(\mathfrak{p})\).

We have \(\theta|_\mathfrak{p} = -\text{id}_\mathfrak{p}\).

The map \(\Theta : G/K \to G/K\) is an isometry, it reverses geodesics through \(eK\).

\(G/K\) is the space of Cartan involutions of \(G\), it is a “symmetric space”.

\(L \subset G\) is compact then \(L\) contains a fixed point \(gK\) so \(L \subset gKg^{-1}\), hence \(K\) is a maximal compact subgroup and all maximal compact subgroups are conjugate.

\[P_n = \{Q \in M_n(\mathbb{R}) : Q \text{ symmetric positive definite with } \det(Q) = 1\}\,.
\]

The space \(\text{SL}_n(\mathbb{R})\) acts on \(P_n\) via \(g \cdot Q = gQg^T\). This action is transitive, the stabilizer of \(I_n\) is \(\text{SO}(n)\) hence \(P_n \cong \text{SL}_n(\mathbb{R})/\text{SO}(n)\). With a \(Q\) we define \(g_Q(X, Y) = \text{Tr}(Q^{-1}XQ^{-1}Y)\).

\(\text{SL}_2(\mathbb{R})\) acts on \(\{x + iy > 0\}\) via \(g\hat{z} = \frac{az + b}{cz + d}\). We have \(y\left(\frac{az + b}{cz + d}\right) = \frac{y(z)}{|cz + d|^2} > 0\) where \(y(z) = \text{Im}(z)\) and \(\text{Stab}_{\text{SL}_2(\mathbb{R})}(i) = \text{SO}(2)\).

More generally, we can define this upper-half plane as
\[\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) : Z \text{ symmetric and } y(Z) = \text{Im}(Z) \text{ is positive definite}\}\,.
\]
Let \( g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{GL}_{2n}(\mathbb{R}) \) try setting \( g \cdot Z = (AZ + B)(CZ + D)^{-1} \)

**Check:** This is \( \mathbb{H}_n \) if and only if \( g \in \text{Sp}_{2n}(\mathbb{R}) \). We get a transitive action of \( \text{Sp}_{2n}(\mathbb{R}) \) on \( \mathbb{H}_n \), the stabilizer of \( iI_n \) is \( U(n) \).

In the 2-dimensional case, we know that we can send the upper-half plane to a circle via the Cayley transform \( z \mapsto \frac{1+z}{1-z} \). Here again we have a corresponding bounded symmetric domain: Open bounded subset \( D \subset \mathbb{C}^n \) where the group of biholomorphism is "large enough". In the general case, it’s also probably the map \( Z \mapsto (I + Z)(I - Z)^{-1} \).

On is there is a canonical metric called Bargmann metric, invariant by \( \text{Aut}(D) \). This with Cauchy formula lets us linking purely geometric volumes to algebraic quantities.