CHAPTER 4

Siegel modular forms

4.1. The symplectic group and the Siegel upper half-space

4.1.1. The symplectic group. Fix a field \( F \) and a finite-dimensional vectorspace \( V/F \). A symplectic form on \( V \) is a non-degenerate bilinear form \([·,·]: V \times V \to F\) such that \([v,v] = 0\) for all \( v \in V \). A symplectic vector space is a pair \((V, [·,·])\) as above.

Exercise 18. Let \([·,·]: V \times V \to F\) be a bilinear form.

1. If the form is symplectic, it is alternating: \([u,v] = -[v,u]\).
2. If \(\text{char } F \neq 2\) and the form is alternating, it is symplectic.

Proof. \([u+v,u+v] = [u,u] + [u,v] + [v,u] + [v,v]\). \(\square\)

Example 19. Let \( U \) be an \( F \)-vectorspace and equip \( V = U \oplus U^* \) with the canonical form \([ (q,p) , (q',p') ] = \langle q',p \rangle - \langle q,p' \rangle \), where the angle brackets denote the pairing between \( U, U^* \).

Concretely, let \( \{u_i\} \subset U \) be a basis, \( \{u^*_i\} \subset U^* \) the dual basis. Then if \( v = \sum_{i=1}^n x_i u_i + \sum_{i=1}^n x_{i+n} u^*_i \) and \( v' = \sum_{i=1}^n y_i u_i + \sum_{i=1}^n y_{i+n} u^*_i \) in \( V \) we have
\[
[v,v'] = t_J’xJy
\]
where \( J = \begin{pmatrix} I_n & 0 \\ -0 & I_n \end{pmatrix} \).

Exercise 20 (Darboux’s Theorem). Show that any symplectic vector space is isomorphic to the canonical example.

Fix a symplectic vector space \( V \).

Lemma-Definition 21. Let \( L \subset V \) be a subspace, maximal under the assumption that \([·,·]|_L = 0\). Then \( \dim V = 2 \dim L \). Such subspaces are called Lagrangian subspaces.

Proof. Consider the map \( V/L \to L^* \) given by the symplectic form. \(\square\)

Lemma-Definition 22. Let \( L \subset V \) be a Lagrangian subspace, and let \( L^* \subset V \) be a subspace, maximal under the assumption that \( L^* \) is linearly disjoint from \( L \) and such that \([·,·]|_{L^*} = 0\). Then \( L^* \) is Lagrangian, \( V = L \oplus L^* \), and the symplectic form induces a non-degenerate pairing between \( L, L^* \). Such Lagrangian subspaces are called dual to \( L \). A representation \( V = L \oplus L^* \) is called a Lagrangian splitting of \( V \).

Notation 23. Given a Lagrangian splitting \( V = L \oplus L^* \) we identify \( L^* \) with the dual of \( L \) via the symplectic form. We use the notation ‘a to denote dual maps with respect to this duality.
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Exercise 24 (Darboux’s Theorem, again). Show that every symplectic vector space is isomorphic to the canonical example.

Definition 25. Let $V$ be a symplectic vector space. The associated symplectic group is the group

$$\text{Sp}(V) = \{ g \in \text{GL}(V) \mid \forall \nu, \nu' \in V : [g \nu, g \nu'] = \nu, \nu' \}.$$ 

The group of symplectic similitudes is

$$\text{GSp}(V) = \{ g \in \text{GL}(V) \mid \exists \lambda(g) \in F^\times \forall \nu, \nu' \in V : [g \nu, g \nu'] = \lambda(g) [\nu, \nu'] \}.$$ 

Exercise 26. Show that (with $2n = \dim_F V$) these are isomorphic to the groups of $F$-points of the linear algebraic groups

$$\text{Sp}_{2n} = \{ g \in \text{GL}_{2n} \mid ^t g J g = J \}$$

$$\text{GSp}_{2n} = \{ g \in \text{GL}_{2n} \mid \exists \lambda(g) \in \text{GL}_{1} : ^t g J g = \lambda(g) J \}.$$ 

Show that $\lambda : \text{GSp}_{2n} \to \text{GL}_1$ is a group homomorphism (and that $\det : \text{GL}_{2n} \to \text{GL}_1$ is the usual determinant).

Notation 27. Fixing a Lagrangian splitting $V = L \oplus L^*$ we may write any $g \in \text{GSp}(V)$ in the form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \in \text{Hom}(L, L)$, $b \in \text{Hom}(L^*, L)$ etc.

Exercise 28. $g \in \text{Sp}_{2n}(V)$ iff $^t ac = ^t ca \in \text{Hom}(L, L^*)$, $^t bd = ^t db \in \text{Hom}(L^*, L)$ and $^t ad - ^t bc = \text{Id}_{L^*} \in \text{Hom}(L^*, L^*)$.

Remark 29. In the standard example, we may think of $a, b, c, d \in M_n(F)$ and $^t$ denoting the usual transpose.

4.1.2. Distinguished subgroups and the affine patch.

Exercise 30 (Darboux’s theorem, yet again). Show that $\text{Sp}(V)$ acts transitively on the set of pairs $(L, L^*)$ of dual Lagrangian subspaces.

Definition 31. The Levi subgroup [of the Siegel parabolic], to be denoted $M$, is the point stabilizer of a pair $(L, L^*)$. It is necessarily a closed subgroup.

Note that we have a natural homomorphism $M \to \text{GL}(L)$ by restriction.

Exercise 32. For $h \in \text{GL}(L)$ let $m(h) = \text{diag}(h, ^t h^{-1}) \in \text{GL}(V)$. Then $m(h) \in \text{Sp}(V)$ and the map $m : \text{GL}(L) \to M$ is an isomorphism.

Lemma-Definition 33. Let $z \in \text{Hom}(L^*, L)$ be symmetric in that $z = ^t z \in \text{Hom}(L^*, L^{**}) = \text{Hom}(L^*, L)$. Then $n(z) = \begin{pmatrix} \text{Id}_L & z \\ z & \text{Id}_{L^*} \end{pmatrix} \in \text{Sp}(V)$ and $N = \{ n(z) \mid z \in \text{Sym}^2 L \}$ is a subgroup of $\text{Sp}(V)$, the unipotent radical [of the Siegel parabolic]. The map $z \mapsto n(z)$ is an isomorphism $(\text{Sym}^2 L, +) \to N$.

Exercise 34. Show that $N$ is normalized by $M$. Show that $P = MN \simeq M \times N$ (the Siegel parabolic subgroup) is the stabilizer of $L$ in the transitive action of $\text{Sp}(V)$ on the set of Lagrangian subspaces of $V$. Show that $N$ a closed subgroup of $P$, hence of $\text{Sp}(V)$.

Proposition 35. Show the set of Lagrangian subspaces is closed in $\text{Gr}(n, V)$. In particular, $\text{Sp}(V)/P$ is a projective variety and $P$ is a parabolic subgroup.
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**Definition 36.** Call a Lagrangian subspace $\tilde{L}$ generic if its projection onto $L^\ast$ via the decomposition $V = L \oplus L^\ast$ is surjective.

**Exercise 37.** A Lagrangian is generic iff it is dual to $L$.

**Lemma 38.** The set of generic Lagrangians is exactly the $N$-orbit of $L^\ast$. It is an open subset of $\text{Sp}(V)/P$ on which $N$ acts freely.

**Proof.** Since $\text{Sp}(V)$ acts transitively on pairs of dual Lagrangians, $P = \text{Stab}_G(L)$ acts transitively on Lagrangians dual to $L$. But $P = NM$ where $M = \text{Stab}_P(L^\ast)$ and the claim follows.

**Proof.** Let $\tilde{L}$ be a generic Lagrangian subspace. Then the inclusion $\tilde{L} \subset V \cong L \oplus L^\ast$ realises $\tilde{L}$ as the graph of a function $z: L^\ast \to L$, and it is clear that $\tilde{L} = t(z)L^\ast$. To show that $t(z) \in \text{Sp}(V)$ we need to verify that $z$ is self-dual. For this note that $t^1z$ is defined by the relation $[t^1z(u), v] = [z(v), u]$ for all $u, v \in L^\ast$. However, $u + z(u), v + z(v)$ both belong to the Lagrangian subspace $\tilde{L}$ and it follows that

$$0 = [u + z(u), v + z(v)]$$

$$= [u, v] + [u, z(v)] + [z(u), v] + [z(u), z(v)]$$

$$= [z(u), v] + [u, z(v)]$$

since $L^\ast$ and $L$ are both Lagrangian. It follows that $[z(u), v] = [z(v), u]$ for all $u, v \in L^\ast$, in other words that $t^1z = z$. The action is free since $t(z)L^\ast \neq L^\ast$ whenever $z \neq 0$.

Finally, it suffices to show that if $V = L \oplus L^\ast$ then $\{ \tilde{L} \in \text{Gr}(n, V) \mid \tilde{L} \cap L = \emptyset \}$ is open.

**Exercise 39.** Let $Z = Z(M)$. Show that $Z \cong \text{GL}_1$ and that $Z_{\text{Sp}(V)}(Z) = M$ (hint: note that $V = L \oplus L^\ast$ is exactly the eigenspace decomposition of $V$ wrt the action of $Z$).

**Exercise 40.** Fix a symmetric isomorphism $I: L^\ast \to L$ (i.e. $t^1I = I$) and let $w = \begin{pmatrix} -I & 1 \\ I & 0 \end{pmatrix}$. Then $w \in \text{Sp}(V)$ normalizes $Z$, on which it acts by the non-trivial automorphism. Further, $w$ exchanges the Lagrangian subspaces $L, L^\ast$.

**Solution.** It is clear that $wL^\ast = L$ and $wL = L^\ast$. Also, $w^2 = -\text{Id}_V$ so $w^{-1} = -w$. If $u \in L$ and $t \in \text{GL}_1$ then

$$\begin{align*}
wm(t)w^{-1}u &= wm(t)(I^{-1}u) = wt^{-1}(I^{-1}u) = t^{-1}II^{-1}u = t^{-1}u = m(t^{-1})u
\end{align*}$$

(since $I^{-1}u \in L^\ast$) and similarly for $v \in L^\ast$, so $wm(t)w^{-1} = m(t^{-1})$. We still need to verify that $[wu, wv] = [u, v]$ for all $u, v \in V$, but it suffices to consider the case $u \in L, v \in L^\ast$ and then

$$[wu, wv] = [-I^{-1}u, Iv] = [-I^{-1}Iv, u]$$

$$= -[v, u] = [u, v].$$

**Lemma 41** (Bruhat decomposition). The “big cell” $NW \subset \text{Sp}(V)$ is open.
shows that every element of \( W \) has invertible lower left corner, and conversely, since \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we see that \( h = -Ic \) and \( z = ac^{-1} \) are uniquely determined, and furthermore that

\[
\begin{pmatrix} 1 & z \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -I^{-1} & I \\ h & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & z' \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -zI^{-1}h & -zI^{-1}hz' + I'h^{-1} \\ -I^{-1}h & -I^{-1}hz' \end{pmatrix}
\]

so, as noted in the lemma, if \( cz + d \) is invertible we have \( gn(z)w \in n(z')wm(h)N \) with \( h = I(cz + d)^{-1}I^{-1} \) and \( z' = (az + b)(cz + d)^{-1} \). Note that \( h \) and \( z' \) are independent of the choice of \( I \).

**Exercise 43.** Show that \((u, v) = [u, wv] \) is a symmetric bilinear map.

**Proof.** \((v, u) = [v, wu] = [w^{-1}v, w] = [-wv, u] = [u, wv] \).

**Lemma-Definition 44 (Maximal tori).** Let \( A \subset GL(L) \) be the set of all matrices diagonal wrt to a basis. Then \( A \) is a maximal abelian subgroup of \( GL(L) \) and \( T = \{ m(a) \mid a \in A \} \) is a maximal abelian subgroup of \( Sp(V) \), the maximal torus.

**Proof.** That \( Z_{GL(L)}(A) = A \) is well known. Next, we have \( Z_G(T) \subset Z_G(Z(M)) = M \) since \( Z(M) \subset T \). It follows that \( Z_G(T) = Z_M(A) = m(\{ Z_{GL(L)}(A) \}) = m(A) = T \).

**Lemma 45.** \( \mathfrak{sp}_{2n} = \text{Lie} \cdot \text{Sp}_{2n} \) \( = \{ X \in M_{2n} \mid ^tJX + JX = 0 \} \). \( \text{Lie} \cdot \text{Sp}(V) \) \( = \{ X \in \text{End}(V) \mid \forall \xi, \xi' \in V : [X\xi, \xi'] + [\xi, X\xi'] = 0 \} \).

**Exercise 46.** \( X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sp}_{2n} \) iff \( d = -t^ac, t^bc = c, ^tb = b \). In particular, \( \dim \mathfrak{sp}_{2n} = 2n^2 + n \).

**Exercise 47.** Let \( \{ e_i \}_{i=1}^{n} : A \rightarrow GL_1 \) be the eigenvalues with respect to our fixed basis of \( W \), thought of as functions \( T \rightarrow GL_1 \). Then the joint eigenvalues \( \alpha : T \rightarrow GL_1 \) acting on \( \text{Lie} \cdot \text{Sp}(V) \) are \( \{ e_i \pm e_j \}_{i \neq j} \cup \{ \pm 2e_i \} \cup \{ 0 \} \). The zero eigenspace is \( \text{Lie} \cdot T \) and each other eigenspace is one-dimensional.

**Solution.** Note that \( \text{Lie} \cdot \text{Sp}(V) = \text{Lie} \cdot M \oplus \text{Lie} \cdot N \oplus \text{Lie} \cdot \tilde{N} \) where \( \tilde{N} = \text{Lie} \cdot wNw^{-1} \), and analyze the action of \( T \) in each case.
4.1.3. Real symplectic spaces and Siegel upper half-space. Suppose now that $V$ is a real symplectic vector space and fix a Lagrangian splitting $V = L \oplus L^*$. Let $G = \text{Sp}(V)$, $G(\mathbb{C}) = \text{Sp}(V \otimes_{\mathbb{R}} \mathbb{C})$. We similar have subgroups $M, M(\mathbb{C}), N, N(\mathbb{C}), P, P(\mathbb{C}), T, T(\mathbb{C})$. Let $w$ be the long Weyl element from the previous section.

Exercise 48. For $\zeta = a + ib \in \mathbb{C}$ and $x \in V$ set $\zeta \cdot x = ax + bwx$. This endows $V$ with the structure of a complex vector space.

Solution. We have $w^2 = -1V$.

Exercise 49. Suppose that $I: L^* \to L$ is negative definite. Then the real-valued pairing $(x, y) = [x, wy]$ is the real part of a hermitian pairing on $V$.

Solution. We already know that this is $\mathbb{R}$-bilinear. To check definiteness let $x = q + p$ with $q \in L$ and $p \in L^*$, in which case

$$(x, x) = [x, wx] = [q + p, Ip - I^{-1}q] = [p, Ip] - [q, I^{-1}q] = -[Ip, p] - [q, I^{-1}p].$$

Finally, $(ix, y) = (wx, y) = [wx, wy] = [x, y]$ is symplectic.

Exercise 50. The unitary group $K$ associated to this Hermitian pairing is a subgroup of $G$.

Solution. The unitary group preserves the complex part of the Hermitian pairing.

Proposition 51. $K$ is a maximal closed subgroup of $G$.

Proof. The representation of $K$ on $\text{sp}V$ decomposes as the direct sum $\text{Lie} K \oplus p$ where $p$ is irreducible, so $K$ is a maximal connected subgroup. It follows that any subgroup containing $K$ is contained in the normalizer of $K$. But if $g \in G$ normalizes $K$ then $g$ maps the inner product $(\cdot, \cdot)$ to another one fixed by $K$. By Schur’s Lemma $g$ is scalar and since $\text{Sp}(V) \subset \text{SL}(V)$ this implies $g = \pm 1V \in K$. □

Corollary 52. Let $Z(K)$ be the centre of the group $K$ (recall that the centre of $U(n)$ is isomorphic to $U(1)$). Then $Z_G(Z(K)) = K$.

Exercise 53. Let $U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$. Then $Z \simeq \text{Hom}(U(1), U(1))$ via the map $n \mapsto (z \mapsto z^n)$ where $\text{Hom}(U(1), U(1))$ is either in the category of compact Lie groups or of real algebraic groups.

Corollary 54. There are exactly two isomorphisms $\rho: U(1) \to Z(K)$.

Exercise 55. There are two eigenspaces $L_\pm$ of $\rho$ in $V_\mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C}$ (on which $U(1)$ acts by its two isomorphic representations). These spaces are Lagrangian, generic with respect to $L_\mathbb{C} \subset V_\mathbb{C}$.

Lemma 56. $\text{Stab}_G(L_+) = K$.

Proof. Since $K$ centralizes its center, it acts on each eigenspace and $K \subset \text{Stab}_G(L_+)$. Equality follows since $K$ is a maximal closed subgroup. □

Definition 57. The image of $G/K$ in $G(\mathbb{C})/P(\mathbb{C})$ as the orbit of $L_+$ is called Siegel upper half space and denoted $\mathbb{H}$.

Lemma 58. Let $g \in G$ and $z \in \mathbb{H}$. Then $cz + d$ is invertible.
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Proof. The Lagrangian \(gL+\) is one of the Lagrangians corresponding to the maximal compact subgroup \(gKg^{-1}\), so it is also generic. □

Proposition 59. \(G/K\) is open in the affine patch \(N_wP/C\).

Proof. \(\dim_R G/K = 2n^2 + n - n^2 = n(n+1)\), \(\dim_R N_C = 2 \dim_C N_C = 2 \binom{n}{2}\) since \(N_C\) is the space of symmetric matrices. □

Corollary 60. \(G/K\) has a complex structure, compatible with its manifold structure.

4.1.4. Vector bundles and factors of automorphy. In terms of the first section, if \(W\) is an \(F\)-vectorspace, any finite-dimensional representation \(\tilde{\sigma}: M \to GL(W)\) induces a vector bundle \(G \times P W \to G/P\). The restriction to the affine patch \(N_w \subset G/P\) is isomorphic to \(N \times W\). Our explicit \(G\)-action then reads:

\[
g \cdot (n(z)wP,\omega) = (n ((az+b)(cz+d)^{-1}) wP, \tilde{\sigma} (I(cz+d)^{-1}I^{-1}))
\]

Returning to the case of real scalars, any finite-dimensional complex representation \((\sigma,W)\) of \(K\) induces the vector bundle \(G \times_K P \to G/K\). The inclusion \(K \subset GL(L_+)^\ast\) of a maximal compact subgroup; by the Weyl unitary trick we can extend \(\sigma\) to a holomorphic representation \(\tilde{\sigma}: GL(L_+) \to GL(W)\), equivalently to a representation \(\tilde{\sigma}: P_C \to GL(W)\).

Proposition 61. The inclusions \(G \times_K W \subset Nw \times W \subset G_C \times P_C\) are compatible with the bundle structures. In particular, \(G \times_K W\) is a holomorphic vector bundle over \(G/K\).