1. Representation of semisimple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra.

Recall : by Lie’s Theorem : If $\mathfrak{g}$ is solvable, and $V$ is a representation of $\mathfrak{g}$ then there is $v \in V$ nonzero such that $v$ is an eigenvector of $X$ for all $X \in \mathfrak{g}$.

A consequence : Let $\mathfrak{g}_{ss} = \mathfrak{g}/\text{Rad}(\mathfrak{g})$ where $\text{Rad}(\mathfrak{g})$ is the maximal solvable subalgebra. Every irrep of $\mathfrak{g}$ is of the form $V_0 \otimes L$ where $V_0$ is an irreducible representation of $\mathfrak{g}_{ss}$ and $L$ is 1-dimensional.

Upshot : we can often pass representations to the semisimple part.

Question : How do we study representations of $\mathfrak{g}$, fix $\mathfrak{g}$ to be semisimple.

Step 1 : Find a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, i.e. a maximal abelian diagonalizable subalgebra of $\mathfrak{g}$.

Step 2 : $\mathfrak{h}$ acts on $\mathfrak{g}$ via adjoint representation, we get the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_{\alpha}.$$ 

Same for representations $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ (weight space decomposition).

We define

- roots of $\mathfrak{g}$ : $\alpha \in \mathfrak{h}^* \setminus \{0\}$ st $\mathfrak{g}_{\alpha} \neq 0$.
- root space : $\mathfrak{g}_{\alpha}$
- $R = \{\alpha \in \mathfrak{h}^* \text{ roots}\}$
- $R$ generate a lattice $\Lambda_R \subset \mathfrak{h}^*$ of rank $\text{dim} \mathfrak{h}$. Call this root lattice.
- If $V$ is a representation then call $\text{dim} V_\alpha$ the multiplicity of $\alpha$
- $\mathcal{I}_\beta : V_\alpha \rightarrow V_{\alpha + \beta}$
All weights of irreps are congruent modulo $\Lambda_R$.

**Step 3:** Find distinguished subalgebra $s_\alpha \simeq \mathfrak{sl}_2 \subset \mathfrak{g}$ for each $\alpha$
set
$$s_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sl}_2.$$ Pick a basis $X_\alpha, Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha]$.

$H_\alpha$ is determined by $X_\alpha, Y_\alpha$ and the requirement that $\alpha(H) = 2$

**Step 4:** A consequence based on representations of $\mathfrak{sl}_2 \mathbb{C}$, any representation of $\mathfrak{g}$ have *integral* eigenvalues at each $H_\alpha$

Define the *weight lattice*
$$\Lambda_\omega = \{ \beta \in \mathfrak{h}^* : \beta(H_\alpha) \in \mathbb{Z} \ \forall \alpha \in R \}.$$ **Step 5:** Account for the symmetry of $\mathfrak{sl}_2 \mathbb{C}$ representations.

For $\alpha \in R$ and $\beta \in \mathfrak{h}^*$ we define
\[
W_\alpha(\beta) = \beta - \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)} \alpha = \beta(H_\alpha) \alpha.
\]

$W_\alpha$ reflects the lines spanned by $\alpha$ through the hyperplane $SL_\alpha = \{ \langle H_\alpha, \beta \rangle = 0 \}$

The *Weyl group* of $\mathfrak{g}$ is
$$\mathcal{W} = \langle W_\alpha : \alpha \in R \rangle$$

**Fact.** Set of weights of any representation and its multiplicities is invariant under $\mathcal{W}$.

**Step 6:** Choose a “direction” in $\mathfrak{h}^*$ i.e. choose a real linear functional $\ell$ on $\Lambda_R$ giving a decomposition $R = R^+ \cup \{ 0 \} \cup R^-$. Define a *highest weight vector* of $V$ (rep of $\mathfrak{g}$) to be an eigenvector $v$ such that $v$ is in the kernel of $\mathfrak{g}_\alpha$ for all $\alpha \in R^+$. The highest weight of $v$ to be othe *highest weight*.

**Fact.**
- Every finite-dimensional representation has a highest weight vector.
- Every finite-dimensional irreducible representation has a unique highest weight up to scalar multiple.
- Subspace $W$ of $V$ generated by application $\mathfrak{g}_\beta$, $\beta \in R^-$ on a highest weight vector is irreducible.
• Every vertex of the convex hull of weights of $V$ is conjugate to a highest weight $\alpha$ under $W$.
• $\alpha(H_\gamma) \geq 0$ for all $\gamma \in R^+$. The locus of these inequalities is a (closed) Weyl chamber.

We get the set of weights of $V$ as erights congruent to a highest weight $\alpha$ modulo $\Lambda_R$ and lie in the convex hull of images of $\alpha$ under $W$.

**Theorem.** For any $\alpha$ in the intersection between the Weyl chamber with $\Lambda_\omega$, there exists a unique finite dimensional irrep $\Gamma_\alpha$ of $\mathfrak{g}$ with highest weight $\alpha$.

We get a bijection

$$\text{Weyl chamber } \cap \Lambda_\omega \longleftrightarrow \text{f.d. irrep of } \mathfrak{g}$$

$$\alpha \mapsto \Gamma_\alpha.$$  

**Question :** how to get multiplicities ?

For $\mathfrak{sl}_n$ :

**Step 1 :** $h = \{ \sum_i a_i H_i, \sum a_i = 0 \}$, $h^* = \mathbb{C}[L_1, \cdots, L_n]/(\sum L_i)$  
$L_i(H_i) = 1$.

**Step 2 :** $F_{ij}$ are eigenvectors with root $L_i - L_j$. Roots $R = \{ L_i - L_j ; i \neq j \}$.

**root lattice :** $\lambda_R = \{ \sigma a_i L_i, \sum a_i = 0, a_i \in \mathbb{Z} \}$

**Step 3 :** $s_{L_i - L_j}$ is generated by $E_{ij}$, $E_{ji}$, and $H_i - H_j$.

**Step 4 :** $\sum a_i L_i \in \Lambda_\omega \Leftrightarrow a_1 \equiv \cdots \equiv a_n \mod \mathbb{Z}$ for all $k, \ell$.

**Step 5 :** $W_{L_i - L_j}$ switches $L_i$ and $L_j$ and fixes everything else. $\mathcal{W} = S_n$.

$R^+ = \{ L_i - L_j ; i < j \}$

$R^- = \{ L_i - L_j ; i < j \}$

Weyl chamber = $\{ \sum a_i L_i : a_1 \geq \cdots \geq a_n \}$

For $\mathfrak{sl}_3$ :

irrep $\Gamma_{a,b}$ weight $aL_1 - bL_3$, $a, b \in \mathbb{N}$.

$V = \mathbb{C}^3$ : eigenvalues $\{ L_1, L_2, L_3 \}$

$\mathcal{S}V^*$ : $\mathcal{S}$ eigenvalues are $\{ -L_1, -L_2, -L_3 \}$.

$\text{Sym}^2(V)$, the eigenvalues are $\{ 2L_i, L_i + L_j \}$.

$\text{Sym}^2(V) \otimes V^*$ eigenvalues $\{ 2L_i - L_j, L_i + L_j - L_k, L_i \}$

$\text{Sym}^2(V) \otimes V^* \rightarrow \text{Sym}^2(V)$
Kernel of this map is $\Gamma_{2,1}$ so $\text{Sym}^2 \otimes V^* = \Gamma_{2,1} \oplus V$.

Back to $\mathfrak{sl}_n \mathbb{C}$.

$V = \mathbb{C}^n$

$V$ has highest weight $L_1$

$\text{Sym}^m V$ has highest weight $mL_1$

$\wedge^m V$ has highest weight $L_1 + \cdots + L_m$.

Irreps of $\mathfrak{sl}_n \mathbb{C}$ are $\Gamma_{a_1, \ldots, a_{n-1}}$ with highest weight $(a_1 + \cdots + a_{n-1})L_1 + \cdots + a_{n-1}L_{n-1}$.

$\Gamma_{a_1, \ldots, a_{n-1}} \subset \text{Sym}^{a_1} V \otimes \text{Sym}^{a_2} \wedge^2 V \otimes \cdots \otimes \text{Sym}^{a_{n-1}} \wedge^{n-1} V$.

**Question:** How do you describe $\Gamma_{a_1, \ldots, a_{n-1}}$?

**Weyl construction**

Let $V$ be an $n$-dimensional $\mathbb{C}$-vector space. Consider the natural action of $S_d$ on $V \otimes^d$.

**Def.** Let $\lambda = \lambda_1 \geq \cdots \geq \lambda_n$ be a partition of $d$

A **Weyl module** or **Weyl construction** associated to $\lambda$ of a $\mathbb{C}$-vector space is

$$S_{\lambda} V := V \otimes^d \otimes_{\mathbb{C} S_d} \check{V}_{\lambda},$$

where $\check{V}_{\lambda}$ is the irrep of $S_d$ associated to $\lambda$ (section 6 of Fulton-Harris).

**Fact:** Any endomorphism $g$ of $V$ lifts to an endomorphism of $S_{\lambda} V$. Look at the character of $S_{\lambda} V$

$$\chi_{S_{\lambda} V}(g) = \text{trace of (image of) } g$$

**Theorem.** $\chi_{S_{\lambda} V}(g) = S_{\lambda}(x_1, \cdots, x_n)$ where $x_i$ are eigenvalues of $g$

$$\dim S_{\lambda} V = S_{\lambda}(1, \cdots, 1) = \prod_{i \neq j} \frac{\lambda_i - \lambda_j + (j - i)}{j - i}$$

Those $S_{\lambda}$ are called **Schur polynomials**.

Brief detour to symmetric polynomials

Write $M_\mu(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} x_\sigma(1)^{\mu_1} \cdots x_\sigma(n)^{\mu_n}$.

For example $M_{311}(x_1, x_2, x_3) = 2x_1^4x_2x_3 + 2x_1x_2^3x_3 + 2x_1x_2x_3^2$.

$H_d = \sum_{1 \leq i_1 \leq \cdots \leq i_d \leq n} x_{i_1} \cdots x_{i_d}$, $E_d = \sum_{1 \leq j_1 < \cdots < j_d \leq k} x_{j_1} \cdots x_{j_d}$.

**Schur polynomials** are a specific basis for the algebra of symmetric functions,
\[ S_\lambda(x_1, \cdots, x_n) = \sum_{T \in \text{SSYT}(\lambda)} x^T \] where \( x^T = x^{T_1} \cdots x^{T_n} \) and \( \text{SSYT}(\lambda) \) is the set of semistandard Young Tableau of shape \( \lambda \) (semistandard = can have repeated numbers).

**Facts.** \( + s_\lambda = \sum_{\mu \leq \lambda} K_{\lambda \mu} m_\mu \).

\( K_{\lambda \mu} \) are Kostka numbers: number of SSYT of shape \( \lambda \) with weight \( \mu \).

- Jacobi bialternant formula:

\[
[X^\mu] = \begin{bmatrix}
X_1^{\mu_1} & \cdots & X_n^{\mu_1} \\
\vdots & \ddots & \vdots \\
X_1^{\mu_n} & \cdots & X_n^{\mu_n}
\end{bmatrix}
\]

Then \( S_\lambda = \frac{\det[X^{\lambda_i-n+1}]}{\det[X^{n-1}]} \). Note that the denominator is a Vandermonde determinant.

Back to \( \mathfrak{sl}_n \).

Try to apply Weyl’s construction on \( V = \mathbb{C}^n \). \( S_\lambda V \) can be seen as a rep of \( \text{SL}_n(\mathbb{C}) \) and get a derived action of \( \mathfrak{sl}_n \).

**Prop.** \( S_\lambda \mathbb{C}^n \) is the irrep of \( \mathfrak{sl}_n \mathbb{C} \) if highest weight \( \lambda_1 L_1 + \cdots + \lambda_n L_n \).

“Proof.” \( S_\lambda = \sum_{\mu \leq \lambda} K_{\lambda \mu} m_\mu \) where \( K_{\lambda \lambda} = 1 \) and \( m_\mu \) corresponds to weight \( \sum \lambda_i L_i \).

**Rem :** \( S_\lambda V \cong S_\mu V \) if and only if \( \lambda - \mu = C \).

So \( \Lambda_{a_1, \cdots, a_{n-1}} \rightarrow S_{a_1 + \cdots + a_{n-1}, \cdots, a_{n-1}} \)

**Cor.** \( \dim \Gamma_{a_1, \cdots, a_{n-1}} = \prod_{1 \leq i < j \leq n} \frac{a_i + \cdots + a_{j-1} + (j-i)}{j-i} \).

**Facts.**

- \( S_\lambda(V) \otimes S_\mu V = \bigoplus_{\nu} C_{\lambda \mu}^{\nu} S_{\nu}(V) \) where \( C_{\lambda \mu}^{\nu} \) is the Littlewood-Richardson coefficient.