2. Geometric Satake equivalence

2.1. Affine Grassmannians

$O = \mathbb{C}[[\varpi]]$

$\mathcal{K} = \mathbb{C}(\varpi))$

$H$ complex algebraic group

$H_O = \mathbb{C}$-group scheme which represents the functor $R \mapsto H(\mathbb{C}[[\varpi]])$ ($= L^+ H$ in Timo’s lecture).

$H_K = \mathbb{C}$-group scheme which represents the functor $R \mapsto H(\mathbb{C}(\varpi))$ ($= LH$ in Timo’s lecture).

From now on $G$ is a complex connected reductive algebraic group

$B$ is a Borel subgroup

$T$ maximal torus

$B^-$ is the opposite Borel subgroup

$N$ = unipotent radical of $B$

$\mathfrak{N}^- = \mathfrak{N}$ unipotent radical of $B^-$

$W$ Weyl group of $(G,T)$

$X^\vee = X_*(T)$ cocharacter of $T$

simple coroots $= \Delta_s^\vee = \Delta_s^\vee(G,B,T) \subset$ positive roots $= \Delta_\vee = \Delta_\vee(G,B,T) \subset \Delta_\vee = \Delta_\vee(G,T)$ coroots of $(G,T)$

$X_\vee^\gamma$: dominant characters

Same for $X \supset \Delta \supset \Delta_+ \supset \Delta_s$

Dominance order on $X^\vee$. $\lambda, \mu \in X^\vee$.

$\lambda \leq \mu \iff \lambda - \mu \in \mathbb{Z}_{\geq 0} \Delta_\vee.$

$\rho = \text{halfsum of positive roots } (\langle \rho, - \rangle : X^\vee \to \frac{1}{2}\mathbb{Z})$
Affine Grassmannian $\text{Gr}_G = ((G_K/G_O)_{et})_{red}$ ind-reduced, ind-proper ind-scheme, ind-(of finite type).

2.2 Decompositions

The embedding $T \subset G$ induces a closed embedding $\text{Gr}_T \to \text{Gr}_G : \varpi^\lambda T_G \mapsto L_\lambda$.

$\text{Gr}_T = \mathcal{X}^{\vee}$ via $\lambda \mapsto \varpi^\lambda T_G$.

**Cartan decomposition.** $\text{Gr}_G = \bigsqcup_{\lambda \in \mathcal{X}^{\vee}} \text{Gr}_G^\lambda$ with $\text{Gr}_G^\lambda = O \cdot L_\lambda$. (smooth locally closed subvariety).

We have $\text{Gr}_G^\lambda = \bigsqcup_{\mu \leq \lambda} \text{Gr}_G^{\mu}$ (proj var with algebraic stratification)

$\dim(\text{Gr}_G^\lambda) = \langle 3p, \lambda \rangle$

$P_\lambda^-$ is a parabolic subgroup of $G$ containing $B^-$ and associated with $\{ \alpha \in \Delta_+ | \langle \lambda, \alpha \rangle = 0 \}$.

Then we have $\text{Gr}_G^\lambda \to G/P_\lambda^-$ via $L_\lambda \mapsto P_\lambda^-$. For $\lambda \in \mathcal{X}^{\vee}_+$ This is a Zariski locally trivial fibration whose fibers are affine spaces.

**Consequences.** $\text{Gr}_G^\lambda$ is simply connected (no nontrivial local systems)

**Bruhat decomposition.** $I \subset G_0$ Iwahori subgroup $\to B \subset G$ via $\varpi \mapsto 0$.

Then $\text{Gr}_G = \bigsqcup_{\lambda \in \mathcal{X}^{\vee}} \text{Gr}_{G,\lambda}$ with $\text{Gr}_{G,\lambda} = I \cdot L_\lambda$ (isom. to an affine space).

For $\lambda \in \mathcal{X}^{\vee}_+$ we have

$$\text{Gr}_G^\lambda = \bigsqcup_{\mu \in W \cdot \lambda} \text{Gr}_{G,\mu}$$

$$\xymatrix{ | \\
V \quad G/P_\lambda^- = \bigsqcup_{w \in W/W_\lambda} B w P_\lambda^- / P_\lambda^- \quad (\mu = w \lambda).}$$

**Iwasawa Decomposition.**

$\text{Gr}_G = \bigsqcup_{\lambda \in \mathcal{X}^{\vee}} S_\lambda$ with $S_\lambda = N_K \cdot L_\lambda$

$= \bigsqcup_{\lambda \in \mathcal{X}^{\vee}} T_\lambda$ with $T_\lambda = N_K^- \cdot L_\lambda$.

Both $S_\lambda$ and $T_\lambda$ are ind-varieties.

$S_\lambda = \bigsqcup_{\nu \in \mathcal{X}^{\vee}, \nu \leq \lambda} S_\nu$

$T_\lambda = \bigsqcup_{\nu \in \mathcal{X}^{\vee}, \nu \geq \lambda} T_\nu$


2.3. The Satake Category

$k$ commutative Noetherian ring of finite global dimension.

**Satake Category.** $\text{Perv}_{G_0}(\text{Gr}_G, k)$ $G_0$-equivariant. $k$ perverse sheaves on $\text{Gr}_G$ with respect to the stratification by $G_0$-orbits.

*Pf here.* $\text{Gr}_G$ is an ind-variety and not a variety. $G_0$ is not of finite type.

One overcomes these difficulties in the following way:

if $X \subset \text{Gr}_G$ is a finite union of $G_0$ orbits, then $X$ is a (proj) variety. Moreover, the $G_0$-action on $X$ factors through the action of $L^+_i G$ for $i > 0$.

**Fact.** The category $\text{Perv}_{L^+_i G}(X, k)$ does not depend on the choice of $i$.

Then we set $\text{Perv}_{G_0}(\text{Gr}_G, k) = \lim_{\longrightarrow X} \text{Perv}_{G_0}(X, k)$ where $X$ runs over finite closed unions of $G_0$-orbits.

**Remark.** If $X_q \subset X_2$, $\text{Perv}_{G_0}(X_1, k) \rightarrow \text{Perv}_{G_0}(X_2, k)$ is fully faithful so there are no subtleties in the colimit. Below we will ignore those subtleties.

2.4 Convolution

We consider the twisted product

$$\text{Gr}_G \tilde{\times} \text{Gr}_G = ((G_K \times \text{Gr}_G)/G_0)_{et, red}$$

we have $m : \text{Gr}_G \tilde{\times} \text{Gr}_G \rightarrow \text{Gr}_G$ induced by $(g, hG_0) \mapsto ghG_0$.

**Prop (Mirkovic - Vilonen).** $m$ is stratified semismall with respect to the stratifications $(\text{Gr}_G \tilde{\times} \text{Gr}_G)_{\lambda, \mu \in \mathfrak{X}_G}$ and $(\text{Gr}_G)_{\lambda \in \mathfrak{X}_G}$.

For $\mathcal{F}, \mathcal{G} \in \text{Perv}_{G_0}(\text{Gr}_G, k)$ we consider $p^*(\mathcal{F})^L \boxtimes_k \mathcal{G} \in \text{Perv}(G_K \times \text{Gr}_G, k)$.

$p : G_K \rightarrow \text{Gr}_G$ projection. This is a $G_0$ equivariant perverse sheaf (for the diagonal $G_0$ action). So by descent there exists a perverse sheaf $\mathcal{F} \boxtimes \mathcal{G}$ on $\text{Gr}_G \tilde{\times} \text{Gr}_G$ whose pullback to $G_K \times \text{Gr}_G$ is $p^* \mathcal{F}^L \boxtimes_k \mathcal{G}$, take

$$\mathcal{F} \ast \mathcal{G} := m_*(\mathcal{F} \boxtimes \mathcal{G})$$

**Facts.**

- Convolution is *associative* (i.e. there exists a canonical isom $(- \ast -) \ast - = - \ast (- \ast -)$ functorial in each entry).
- The object $\delta_{G_0} :=$ skyscraper sheaf at $L_0 \in \text{Gr}_G$ is a unit object (i.e. there are canonical isom $\delta_{G_0} \times - \simeq \text{id}$, $d \simeq - \ast \delta_{G_0}$)

So it is a *monoidal* category.
2.5. Statement

\( G_k^\vee = \text{“Langlands dual reductive } k\text{-group”} = \text{Spec}(k) \times_{\text{Spec}(\mathbb{Z})} G_{\mathbb{Z}}^\vee \) where \( G_{\mathbb{Z}}^\vee \) is the unique split reductive group over \( \mathbb{Z} \) whose base change to \( \mathbb{C} \) whose root datum is \( (X^\vee, X, \Delta^\vee, \Delta) \) (exchange roots and coroots).

\[ \text{Rep}(G_k^\vee) = \text{cat of algebraic } G_k^\vee\text{-modules (}O(G_k^\vee)\text{-comodule}) \text{ which are finitely generated as } k\text{-modules}. \]

**Theorem.** There exists an equivalence of monoidal categories \( \text{Perv}_{G_0}(\text{Gr} G, k, \star) \cong (\text{Rep}(G_k^\vee), \otimes) \) under which the forgetful functor \( \text{Rep}(G_k^\vee) \to \text{Mod}_{fg}^k \) corresponds to \( H^*(\text{Gr} G, -) : \text{Perv}_{G_0}(\text{Gr} G, k) \to \text{Mod}_{fg}^k \).

**Remarks.**

- (1.1) Simple objects (in case \( k \) is an algebraically closed field) in \( \text{Rep}(G_k^\vee) \) are classified by highest weights (in \( X_+^\vee \)).
- (1.2) In \( \text{Perv}_{G_0}(\text{Gr} G, k) \): classified by pairs \( (\text{Gr} \lambda G, L) \). Here \( L \) must be \( k \).
- (2) Assume further that \( \text{char}(k) = 0 \). Then we will see later that \( \text{Perv}_{G_0}(\text{Gr} G, k) \) is semisimple. The same is true for \( \text{Rep}(G_k^\vee) \).
- (3) We will do slightly better. We will construct a group scheme \( \tilde{G}_k \) for any \( k \) and an equivalence \( \text{Perv}_{G_0}(\text{Gr} G, k) \cong \text{Rep}(\tilde{G}_k) \) such that \( \tilde{G}_{k'} \cong \text{Spec}(k') \times_{\text{Spec}k} \tilde{G}_k \) for any \( k \to k' \) and show that \( \tilde{G}_{\mathbb{Z}} \) is split reductive (with a canonical maximal torus) with appropriate root datum.

### 2.6. Commutativity

The tensor product in \( \text{Rep}(G_k^\vee) \) is commutative, i.e. for \( M, N \in \text{Rep}(G_k^\vee) \) we have a canonical isomorphism \( M \otimes_k N \cong N \otimes_k M \) so if the theorem is true, the same should hold for \( \text{Perv}_{G_0}(\text{Gr} G, k) \).

In fact the proof will require to construct such an isomorphism before proving the theorem.

**Idea of the construction. (Drinfeld)** Use the moduli representation of \( \text{Gr} G \).

We set \( G = \mathbb{A}_k^1 \), \( C^\infty = \mathbb{Z}_k^1 \setminus \{0\} \). Recall from Timo’s lecture that \( \text{Gr} G \) represents the functor \( R \mapsto \{ (E, \beta) | E \text{ G-bundle on } C_R = C \times \text{Spec} R \beta : E^0_{C^\infty_R} \to E|_{C^\infty_R} \} / \cong \).

**“Global” version.** \( \text{Gr}_{G,G} \to C \) ind-scheme which represents the functor \( R \mapsto \{ (y, E, \beta) | y \in C(\mathbb{R}) E \text{ G-bundle on } C_{\mathbb{R}} \beta : E^0_C \to E|_{C_R} \} \).

But one can do that also over \( C^2 \) : one gets the Beilinson-Drinfeld Grassmannian \( \text{Gr}_{G,G^2} \) : ind-scheme over \( C^2 \) which represents
\[ R \mapsto \left\{ (y_1, y_2, \mathcal{E}, C) \mid y_1, y_2 \in C(R) \mathcal{E} \text{ } G\text{-bundle over } C_R \beta : \mathcal{E}_{C_R \setminus (\Gamma_{y_1} \cup \Gamma_{y_2})} \xrightarrow{\sim} \mathcal{E}_{C_R \setminus (\Gamma_{y_1} \cup \Gamma_{y_2})} \right\}. \]

**Facts.**

1. This functor is represented by and ind-proper ind-scheme over \( C^2 \)
2. \( \text{Gr}_{G, C^2} \times_{C^2} \Delta C = \text{Gr}_{G, C} \times_{C} \Delta C = \text{Gr}_{G} \times \Delta C \)
3. \( \text{Gr}_{G, C^2} \times_{C^2} (C^2 \setminus \Delta C) = (\text{Gr}_{G, C} \times \text{Gr}_{G, C})|_{C^2 \setminus \Delta C} \simeq \text{Gr}_{G} \times \text{Gr}_{G} \times (C^2 \setminus \Delta C) \).

To \( \mathcal{E} \) one associates the pair \((\mathcal{E}_1, \mathcal{E}_2)\) where \( \mathcal{E}_i \) is the \( G \)-bundle obtained by glueing \( \mathcal{E}_{C_R \setminus \Gamma_{y_i}} \) using the trivialization \( \beta (j \neq i) \).

We set \( i : \text{Gr}_{G} \times \Delta C \to \text{Gr}_{G, C^2} \) (closed embedding)
\( j : \text{Gr}_{G} \times \text{Gr}_{G} \times C^2 \setminus \Delta C \to \text{Gr}_{G, C^2} \) (open embedding)

**Theorem (Belinson - Drinfeld).** There exists an isomorphism

\[ i^* \left( j_! P^0 \left( \mathcal{F}_1^L \boxtimes_k \mathcal{F}_2^L \boxtimes_k \mathcal{E}_{C^2 \setminus \Delta C}[2] \right) \right)^{[-1]} \simeq (\mathcal{F}_1 \times \mathcal{F}_2)^L \boxtimes_k \mathcal{E}_{\Delta C^2}[1]. \]

The construction on the left handside is called the fusion product.

**Application.** We have \( \sigma : \text{Gr}_{G, C^2} \xrightarrow{\sim} \text{Gr}_{G, C^2} \) obtained by switching \( y_1 \) and \( y_2 \).
Restrict trivially to \( \Delta C \) and to \((gG_0, hG_0, y_1, y_2) \mapsto (hG_0, gG_0, y_1, y_2)\) over \( C^2 \setminus \Delta C \). Using the fact that \( i^* \simeq i^* \sigma^* \) (because \( \sigma_1 = i \)) one obtains a canonical isomorphism \( \mathcal{F}_1 \ast \mathcal{F}_2 \xrightarrow{\sim} \mathcal{F}_2 \ast \mathcal{F}_1 \).

**Remark.** On fact one needs to twist this isomorphism by a sign depending on the connected component supporting \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) to get the actual commutativity contraint.