Geometric Satake equivalence

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References.
- Mirkovic–Vilonen “Geometric Langlands duality and representations of algebraic groups over commutative rings”.
- Baumann–R. “Notes on the geometric Satake equivalence”

Plan.
- Lecture 1: Constructible sheaves.
- Lecture 2: Statement of the equivalence
- Lecture 3: Proof (for general coefficients)
- Lecture 4: End of the proof tilting modules and parity sheaves

1. Brief reminder on constructible sheaves.

Ref.
- Kashiwa–Schapira “Sheaves on manifolds”
- P. Achar Lecture notes on perverse sheaves
- Chiss- Ginzburg “Rep theory and complex geometry”

1.1 Constructible derived category.

$X$ complex algebraic variety.

Def. An algebraic stratification of $X$ is a finite partition

$$X = \sqcup_{s \in S} X_s$$

with

- (1) Each $X_s$ is a smooth connected locally closed algebraic subvariety of $X$.
- (2) For all $s \in S$, $\overline{X_s}$ is a union of $X_i$’s
(3) Technical condition (“existence of stratified slices”)
c.f. [CG Def 32.23].

\( k \) commutative Noetherian ring of finite global dimension (e.g. field, \( k = \mathbb{Z} \))

**Rk.** Assumptions ensure that

\[
\text{RHom}_k(-, k) : (\mathcal{D}^b\text{Mod}_k^{fg})^{\text{op}} \to \mathcal{D}^b\text{Mod}_k^{fg}.
\]

\( \text{Sh}(X, k) \) : abelian category of sheaves of \( k \)-modules on \( X \) (with respect to the classical topology).

If \( X = \sqcup_{s \in S} X_s \) is an algebraic stratification, we denote by \( i_s : X_s \to X \) the embedding. Then \( F \in \mathcal{D}^b(\text{Sh}(X, k)) \) is said to be \( \mathcal{S} \)-constructible if for all \( s \in \mathcal{S} \) and all \( j \in \mathbb{Z} \) we have \( \mathcal{H}^d(i_s^* F) \) is a local system (= locally constant sheaves with finitely generated stalks).

\( \mathcal{D}^b_S(X, k) = \) full subcategory of \( \mathcal{D}^b\text{Sh}(X, k) \) whose objects are the \( \mathcal{S} \)-constructible complexes. (triangulated subcategory).

**Remak.** The technical condition on stratification is there to ensure that \( \mathcal{D}^b_S(X, k) \) is stable under Verdier duality

\[
\mathbb{D}_X = \text{RHom}_k(-, \omega_X).
\]

**Constructible** derived category \( \mathcal{D}^b_c(X, k) : \) full subcategory of \( \mathcal{D}^b\text{Sh}(X, k) \) whose objects are the complexes \( F \) such that there exists an algebraic stratification \( \mathcal{S} \) such that \( F \) is \( \mathcal{S} \)-constructible.

Again, this is a triangulated subcategory of \( \mathcal{D}^b\text{Sh}(X, k) \).

1.2. Operations on constructible complexes.

\( f : X \to Y \) morphism of algebraic varieties, then we have triangulated functors

\[
\begin{align*}
&f_* : \mathcal{D}^b_c(X, k) \to \mathcal{D}^b_c(Y, k) \\
&f^* : \mathcal{D}^b_c(Y, k) \to \mathcal{D}^b_c(X, k)
\end{align*}
\]

Verdier duality : \( \mathbb{D}_X := \text{RHom}(-, \omega_X) \), where \( \omega_X = \partial^!_{\text{pt}} \partial_! \) with \( \partial : X \to \text{pt} \).

Derived tensor product : \( - \otimes^L_k - : \mathcal{D}^b_c(X, k) \times \mathcal{D}^b_c(X, k) \to \mathcal{D}^b_c(X, k) \).

**Main properties.**

- **Compatibility with convolution.**

\[
\begin{align*}
&(f \circ g)_* = f_* \circ g_* \\
&(f \circ g)^* = g^* \circ f^*
\end{align*}
\]
• **Adjunctions.** \((f^*, f_*)\) and \((f_!, f_!)\) are adjoint pairs.

• **Special cases.** If \(f\) proper then \(f_* = f_!\). If \(f\) is smooth of relative dimension \(n\) then \(f_! = f^*[2n]\)

• **Verdier duality.** \(\mathbb{D}_X \circ \mathbb{D}_X \simeq \text{id}\)

• **Base change.**

Insert cartesian square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g'} & & \downarrow{g} \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

Then we have \((f')^* \circ g' \simeq (g')_! \circ f^*\) and \((f')_! \circ g_* \simeq (g')_* \circ f_!\).

• **Glueing** \(j): U \to X\) open embedding and \(i: Z \to X\) closed embedding with \(X = U \sqcup Z\).

Then

• \(i_* = i!, j_* = j_!\) are fully faithful

• We have functorial distinct triangles

\[
\begin{align*}
& j_* j^* \xrightarrow{\text{adj}} \text{id} \xrightarrow{\text{adj}} i_* i^* \xrightarrow{[1]} \\
& i_* i^* \xrightarrow{\text{adj}} \text{id} \xrightarrow{\text{adj}} j_* j^* \xrightarrow{[1]}
\end{align*}
\]

1.3. **Perverse sheaves (for middle perversity)**

As before, \(k\) commutative Noetherian ring of finite global dimension, \(X = \bigsqcup_{s \in S} X_g\) algebraic stratification.

**Def.**

\[
pD_{\geq 0} = \{ F \in D^b_S(X, k) | \forall s \in S, \ i_s^! F \in D^{\geq \dim X_s}(X, k) \}
\]

\[
pD_{\leq 0} = \{ F \in D^b_S(X, k) | \forall s \in S, \ i_* F \in D^{\leq \dim X_s}(X, k) \}
\]

Define \(\text{Perv}_S(X, k) = pD_{\geq 0} \cap pD_{\leq 0}\).

**Theorem.** \((pD_{\geq 0}, pD_{\leq 0})\) is a bounded \(t\)-structure on \(D^b_S(X, k)\). In particular, \(\text{Perv}_S(X, k)\) is an abelian category, and the exact sequences in \(\text{Perv}_S(K, k)\) are obtained in distinct triangles in \(D^b_S(X, k)\) all of whose vertices belong to \(\text{Perv}(X, k)\) by forgetting the last arrow

\[
F \to G \to H \xrightarrow{[1]},
\]
where $F, G, H$ are perverse.

**Very useful fact.** For $G, G \in \text{Perv}_S(X, k)$

$$\text{Ext}^1_{\text{Perv}_{S}(X, k)}(F, G) \cong \text{Hom}_{\mathbb{D}^b_{S}(X, k)}(F, G[1])$$

This is not true for higher $\text{Ext}$’s.

**Intersection cohomology complexes.** $s \in S$, $d$ local system on $X_s$.

**Claim.** There exists a unique object $IC(X_s, L) \in \text{Perv}_S(X, k)$ such that

- $IC(X_s, L)$ is supported on $\overline{X}_s$
- $s^*IC(X_s, L) = L[\dim X_s]$
- For all $t \in S$ such that $X_t \subset \overline{X}_s$ and $t \neq s$ we have
  $$i_t^! IC(X_s, L) \in D_{< - \dim(X_t)}(X_t, k)$$
  $$i_t^* IC(X_s, L) \in D_{> - \dim(X_t)}(X_t, k).$$

We have natural maps

$$p^H_0(i_{s!}(L[\dim X_s])) \rightarrow p^H_0(i_{s*}L[\dim X_s])$$

factoring through $IC(X_s, L)$ by surjection, then injection.

**Theorem.** Assume that $k$ is a field. We have a bijection

$$\{(s, L)| s \in S \text{ L simple local system on } X_s\}/\text{isom} \cong \{\text{simple objects in } \text{Perv}_S(X, k)\}/\text{isom}$$

$$(s, L) \mapsto IC(X_s, L).$$

**Remark.** If $k$ is a field, then $\mathbb{D}_X IC(X_s, L) \cong IC(X_s, L^\vee)$. In particular, $\mathbb{D}_X$ restricts to an equivalence

$$\text{Perv}_S(X, k)^{op} \cong \text{Perv}_S(X, k).$$

This is not true for general coefficient (already for $X=$point).

**Ex.** If $X_s$ is smooth then $IC(X_s, k) \approx k_{\overline{X}_s}[\dim X_s]$.

**1.4. Stratified semismallness**

$X = \bigsqcup_{s \in S} X_s$, $Y = \bigsqcup_{t \in T} Y_t$ algebraic variety with algebraic stratification, $f : Y \to X$ proper such that

1. For all $t \in T$, $f(Y_t)$ is a union of strata.
• (2) For all $t \in T$ such that $X_s \subset f(Y_t)$ for all $x \in X$, we have
\[
\dim(f^{-1}(X_s) \cap Y_t) \leq \frac{1}{2}(\dim Y_t - \dim X_s)
\]

• (3) For all $t \in T$ for all $s \in S$ such that $X_s \subset f(Y_t)$ then the map $Y_t \cap f^{-1}(X_s) \to X_s$ induced by $f$ is a Zariski locally trivial fibration.

**Proposition.** In this setting, if $f \in \text{Perf}_T(Y, k)$ then $f_*F = f_!F$ belongs to $\text{Perv}_S(X, k)$.

### 1.5. Equivariant perverse sheaves

$X = \bigsqcup_{s \in S} X_s$ algebraic variety with algebraic stratification.

$H$ connected complex algebraic group acting on $X$ with each $X_s$ $H$-stable we have two maps

\[ H \times X \xrightarrow{a \text{ action}} X \quad \xrightarrow{p \text{ projection}} X. \]

**Def.** $F \in \text{Perv}_S(X, k)$ is $H$-equivariant if $a^*F \sim p^*F$.

$\text{Perv}_{S,H}(X, k)$ full subcat of $\text{Perv}_S(X, k)$ whose objects are $H$-equivariant perverse sheaves.

**Facts.**

• (1) $\text{Perv}_{S,H}$ is an abelian subcategory, stable under subquotients (but not under extensions in general).

• (2) If $\mathcal{L}$ $H$-equivariant local system on $X_s$, then $IC(X_s, \mathcal{L})$ is $H$-equivariant.

• (3) $\text{Perv}_H(X, k)$ is the heart of the perverse $t$-structure on the $H$ equivariant $S$-contractible derived category of Bernstein-Lunts.

• (4) If $X \to Y$ is a (Zariski locally trivial) $H$-torsor and $\mathcal{S}$ is the pullback of the stratification $\mathcal{T}$ on $Y$, then the category $\text{Perv}_{S,H}(X, k) \xrightarrow{X \to Y} \text{Perv}_T(Y, \mathcal{T})$.

### 1.6. Parity complexes

$X = \bigsqcup_{s \in S} X_s$ algebraic variety with algebraic stratification.

**Assumptions :**

• $k$ is a field

• For any $s \in S$ all local systems on $X_s$ are trivial (i.e. the fundamental groups of $X_s$’s are trivial)
- For all \( s \in S \) we have \( H^{\text{odd}}(X_s; k) = 0 \).

**Def.** \( \mathcal{F} \in D^b_S(X, k) \) is called *even* (resp. *odd*) if \( H^{\text{odd}}(\mathcal{F}) = H^{\text{odd}}(\mathcal{D}_X \mathcal{F}) = 0 \) (resp. \( H^{\text{even}}(\mathcal{F}) = H^{\text{even}}(\mathcal{D}_X \mathcal{F}) = 0 \)).

\( \mathcal{F} \) is called *parity* if it is a direct sum of an even object and an odd object.

**Exercise.** If \( \mathcal{F} \) is even and \( \mathcal{G} \) is odd, then

\[
\text{Hom}_{D^b_S(X, k)}(\mathcal{F}, \mathcal{G}) = 0.
\]

**Theorem.** (Juteau – Mautner– Williamson) For any \( s \in S \) there exists at most one indecomposable object parity complex \( \mathcal{E}_s \in D^b_S(X, k) \) supported on \( \overline{X_s} \) and such that \( i_*^\text{S}_s \mathcal{E}_s \twoheadrightarrow k[\dim X_s] \). Moreover, any indecomposable parity complex is of the form \( \mathcal{E}_j[n] \) for some \( s \in S \) and \( n \in \mathbb{Z} \), and any parity complex is a direct sum of indecomposable parity complexes.

**Remark.** It can happen that \( \mathcal{E}_s \) does not exist. But it always exists for \( X \) affine Grassmannian (with the stratification by orbits of a parahoric subgroup), or for partial flag varieties of Kac-Moody groups.