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$F$ is a $p$-adic field.

$G$ is a split connected reductive group over $F$. $G = G(F)$

“Spherical part of the spectrum of $G(F)$”.

**Analogy.** Fourier transform on $\mathbb{R}$.

From $f$, a function on $\mathbb{R}$ (Schwarz, ... ) we get $\hat{f}$ on $\hat{\mathbb{R}} \cong \mathbb{R}$ (the isomorphism isn’t canonical), where $\hat{\mathbb{R}}$ is $\{ \psi_x : \psi_x(y) = e^{2\pi i xy} | x \in \mathbb{R} \}$

(Fourier inversion) $f(x) = (\text{constant}) \int_{\hat{\mathbb{R}}} \hat{f}(\Psi) \overline{\Psi(x)} \, d\psi$.

Important: $d\psi$ is a measure on $\hat{\mathbb{R}}$, what allows us to compute it is that $dx = d\psi$.

$\hat{f}(\psi_x) = "\text{Fourier coefficient of } f \text{ at } x"$

Our group is $G(F)$, the analogy gives all (unitary?) reps of $G(F)$, too complicated.

If we look at the smaller subset of spherical representations, i.e. $\pi$ having a $K$-fixed vector ($K = G(O_F)$). This smaller subset has a 1-1 correspondence with bi-$K$-invariant functions, which we expect to be “orthogonal” to the part of the spectrum without fixed vectors. (Not quite true, but true replacing $K$ by $I$)

What is true: We can recover such a function $f$ from its “Fourier coefficients” at spherical representations (= Satake transform of $f$).

$f \in \mathcal{H}(G/K) \to S(f)(\pi) = \int_G f(g)(\text{matrix coeff of } \pi)d\!g = "\hat{f}(\pi)"$.

Know: Spherical representations : $\pi = \text{ind}_B^G(| \cdot |_{p_i}^{s_i} \times \cdots \times | \cdot |_{p_n}^{s_n})$, $s_i \in \mathbb{C}$. Here $\text{ind} = \text{Ind}() \otimes \delta_B^{-1/2}$. When this is reducible, has a unique spherical subquotient (by rearranging $\{s_i\}$, can force it to be a quotient).

Let $(z_i = |\omega|^{s_i}) \in (\mathbb{C}^\times)^n/W$, identify $\pi$ as $\pi_{(z_1, \ldots, z_n)}$. 
For Fourier transform of bi-$K$-invariant function can be found just with the spherical part of representations via Satake transform, because it’s an isomorphism, don’t need the rest of reps!

So $S(f)$ is a function on $(\mathbb{C}^\times)^n/W \leftrightarrow W$-invariant functions on $(\mathbb{C}^\times)^n$, it is actually regular! (i.e. $W$-invariant polynomial in $z_i^{\pm 1}$).

We got our formula $S(f)(\pi_z) = \int_G f(g) E_z(g) \, g$

$z \in (\mathbb{C}^\times)^n/W$, $E_z(g)$ is a bi-$K$-invariant matrix coefficient such that $E_z(1) = 1$, it is a spherical function.

If we believe that $S(f)$ is a polynomial in $z_i^{\pm 1}$, get $S(f) \in \mathbb{C}[\hat{T}/W]$, $\hat{T}$ = Langlands dual torus of $T \subset G$ i.e. $X^*(\hat{T}) = X_*(T)$.

$T(\mathbb{C}) = X_*(T) \otimes_{\mathbb{C}} \mathbb{C} = X^*(T) \otimes \mathbb{C}$.

Relation to Langlands

(packets of) representations of $G(F)$ have a correspondence with Langlands parameters $\sigma :$ Weyl-Deligne group of $F \to L^G$. For spherical : “Deligne” is not relevant. Weil group of $F$.

$1 \to I_F \to \text{Gal}(\overline{F}/F) \to \langle \text{Frob} \rangle \to 1.$

The term on the right is the Galois group of a maximal unramified extension. “Unramified” here can be seen as $\sigma|_{I_F} = 1$.

$T \subset G \leftrightarrow (X^*(T), \phi, X_*(T), \phi^\vee).$ The group $L^G(\mathbb{C})$ is a complex Lie group with max torus $\hat{T}$, root system $\phi^\vee$. $\sigma(\text{Frob}) \in \hat{T}(\mathbb{C})/W$.

**Theorem.** Satake transform is an isomorphism $\mathcal{H}(G//K) \to \mathbb{C}[\hat{T}/W]$.

Thomas proved surjectivity.

Injectivity : closely related to “Plancherel formula”.

On $G$ : $(f_1, f_2) = \int_G f_1(g) \overline{f_2(g)} \, dg = \int_{R(G)} \hat{f}_1 \int_{f_2} \, d\pi$ ( $\hat{\pi}$ is the contragradient rep). Plancherel measure on $R(G)$ = space of unitary reps of $G$.

The Plancherel measure on $\hat{T}(\mathbb{C})/W$ would give $f_{1,2} \in \mathcal{H}(G//K)$, $(f_1, f_2) = \int_{T/W} S(f_1) \overline{S(f_2)} \, d\mu^\text{Pl}(z_1, \cdots, z_n)$.

**Question.** Plancherel measure on $\hat{T}(\mathbb{C})/W$?

The algebra $\mathbb{C}[\hat{T}/W] \cong \mathbb{C}[X^*(\hat{T})]$:

- Natural generators (as an algebra):
\[\xi_1, \cdots, \xi_n = \text{“elementary symmetric polynomials” (they can be seen as the characters of irreducible representations corresponding to the fundamental weights).}\]

- Bases as \(\mathbb{C}\)-vector space \((=W\text{-invariant polynomial functions on } \hat{T}(\mathbb{C}) = (\mathbb{C}^\times)^n)\).

One natural basis: \(\mu \in X^*(\hat{T}) \rightarrow e^\mu\), a formal symbol. Elements of \(\mathbb{C}[X^*(\hat{T})]\) (convolution ring, with this notation it is product of polynomials) are of the form \(\sum_{\mu} c_\mu e^\mu : t \mapsto \sum_{\mu} c_\mu \mu(t)\) (finitely many nonzero terms).

- The first natural basis for \(\mathbb{C}[X^*(T)]^W\) is \(\sum_{\mu \in W(\lambda)} e^\mu\), where \(W(\lambda)\) is the orbit of \(\lambda\) under \(W\).

- The second basis: \(\tau_\lambda =\) character of an irreducible representation of \(L^G\) of highest weight.

- \(S(f_\mu), f_\mu\) character function of \(K\left(\begin{array}{ccc} \varpi_1 & & 0 \\ & \ddots & \\ 0 & & \varpi_n \end{array}\right)K\) (double coset corresponding to \(\mu = (\mu_1, \cdots, \mu_n) \in X^*(T) = X^*(\hat{T})\).

Macdonald’s formula gives a way to give a change of basis from \(\{S(f_\mu)\}\) to \(\{\tau_\mu\}\).