EXERCISES 01 FOR MATH 603D

DUE 5 OCTOBER 2018

Do as many as you can. For the most part, these exercises are things I'd like to have covered in lecture, but didn't want to take the time over. If a question is marked with *, I don't know how to do it.

(1) Let $\phi : R \to S$ be a ring homomorphism and suppose S is f.g. projective when considered as a left R-module by means of ϕ, that is, when R acts on S according to

$$r \cdot s = \phi(r)s.$$

Let M be a left S module, and let R act on M by $r \cdot m = \phi(r)m$. Denote the resulting R module by ${}_R M$. Show that if M is f.g. projective over S, then ${}_R M$ is f.g. projective over R. Construct a "wrong way" map $\phi^! : K_0(S) \to K_0(R)$ by means of $\phi^![M] = [{}_R M]$.

(2) Let R be a ring, and suppose $P \oplus Q = F$ where F is a flat left R-module. Prove that P is a flat left R-module.

(3) Let R be a commutative ring. Let $\mathfrak{a}$ be a finitely generated idempotent ideal, so $\mathfrak{a}^2 = \mathfrak{a}$. Prove that $\mathfrak{a}$ is principal, and generated by an idempotent element.

(4) Let R be a (possibly noncommutative) associative unital ring in which $1 \neq 0$. Prove the following are equivalent:
   (a) R has a unique maximal left ideal;
   (b) R has a unique maximal right ideal;
   (c) There is a unique (two-sided) ideal $\mathfrak{m} \subset R$ such that $R/\mathfrak{m}$ is a division ring[1]
   (d) For any element $x \in R$, at least one of x or $1 - x$ is a unit.

(5) Let k be a field and let $k^{\mathbb{N}}$ denote a countably-infinite dimensional k vector space. Let $R = \mathrm{End}_k(k^{\mathbb{N}})$ denote the ring of k-linear endomorphisms of $k^{\mathbb{N}}$, where multiplication is given by composition. Prove that $R \oplus R \cong R$ as left R-modules, and hence that R does not have the invariant basis property.

(6) Let R be a commutative ring. View $\mathrm{Spec}\, R$ as a topological space. Let $\{X_i\}_{i \in I}$ denote the set of connected components of $\mathrm{Spec}\, R$. Fix a f.g. projective R module Q. Prove that for each X_i, the function $\mathrm{rank}_{\mathfrak{p}}\, Q$ is constant as $\mathfrak{p}$ ranges over the prime ideals in X_i.

(7) Prove the "geomeric Nakayama lemma": let R be a commutative ring and M a f.g. R module. Suppose $\mathfrak{p}$ is a prime ideal of R such that $M_{\mathfrak{p}} = 0$. Show that there exists some element $f \in R \setminus \mathfrak{p}$ such that $M_f = 0$.

(8) Give an example, with proof, of a space X such that $\mathbf{Vect}_{\mathbb{R}}(X)$ is not an abelian category. Prove that $P(\mathbb{Z})$ is not an abelian category.

(9)* Does there exist a normal Hausdorff (necessarily noncompact) topological space X such that Swan's theorem fails for X?

Fun question: We know that $R = \mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain that is not a PID, hence it must have infinitely many maximal ideals. Let $\mathfrak{m}$ be a maximal ideal of R, prove that $R/\mathfrak{m}$ is a finite field. Prove that

[1] the original version of this condition was misstated, in particular the word "unique" was missing, but it was also confusing in other ways.

for each prime number p, there is at least one maximal ideal $\mathfrak{m}$ for which $R/\mathfrak{m}$ has characteristic p. Deduce there are infinitely many prime numbers.