Do as many as you can. For the most part, these exercises are things I’d like to have covered in lecture, but didn’t want to take the time over. If a question is marked with *, I don’t know how to do it.

1. Let $\phi: R \rightarrow S$ be a ring homomorphism and suppose $S$ is f.g. projective when considered as a left $R$-module by means of $\phi$, that is, when $R$ acts on $S$ according to $r \cdot s = \phi(r)s$.

   Let $M$ be a left $S$ module, and let $R$ act on $M$ by $r \cdot m = \phi(r)m$. Denote the resulting $R$ module by $R^M$. Show that if $M$ is f.g. projective over $S$, then $R^M$ is f.g. projective over $R$.

   Construct a “wrong way” map $\phi^!: K_0(S) \rightarrow K_0(R)$ by means of $\phi^![M] = [R^M]$.

2. Let $R$ be a ring, and suppose $P \oplus Q = F$ where $F$ is a flat left $R$-module. Prove that $P$ is a flat left $R$-module.

3. Let $R$ be a commutative ring. Let $a$ be a finitely generated idempotent ideal, so $a^2 = a$. Prove that $a$ is principal, and generated by an idempotent element.

4. Let $R$ be a (possibly noncommutative) associative unital ring in which $1 \neq 0$. Prove the following are equivalent:

   (a) $R$ has a unique maximal left ideal;
   (b) $R$ has a unique maximal right ideal;
   (c) $R$ has a two-sided ideal that is maximal as a left ideal.
   (d) For any element $x \in R$, at least one of $x$ or $1 - x$ is a unit.

5. Let $k$ be a field and let $k^\aleph_0$ denote a countably-infinite dimensional $k$ vector space. Let $R = \text{End}_k(k^\aleph_0)$ denote the ring of $k$-linear endomorphisms of $k^\aleph_0$, where multiplication is given by composition. Prove that $R \oplus R \cong R$ as left $R$-modules, and hence that $R$ does not have the invariant basis property.

6. Let $R$ be a commutative ring. View Spec $R$ as a topological space. Let $\{X_i\}_{i \in I}$ denote the set of connected components of Spec $R$. Fix a f.g. projective $R$ module $Q$. Prove that for each $X_i$, the function $\text{rank}_R Q$ is constant as $p$ ranges over the prime ideals in $X_i$.

7. Prove the “geometric Nakayama lemma”: let $R$ be a commutative ring and $M$ a f.g. $R$ module. Suppose $p$ is a prime ideal of $R$ such that $M_p = 0$. Show that there exists some element $f \notin R \setminus p$ such that $M_f = 0$.

8. Give an example, with proof, of a space $X$ such that $\text{Vect}_R(X)$ is not an abelian category. Prove that $P(Z)$ is not an abelian category.

9. Does there exist a normal Hausdorff (necessarily noncompact) topological space $X$ such that Swan’s theorem fails for $X$?

Fun question: We know that $R = \mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain that is not a PID, hence it must have infinitely many maximal ideals. Let $m$ be a maximal ideal of $R$, prove that $R/m$ is a finite field. Prove that for each prime number $p$, there is at least one maximal ideal $m$ for which $R/m$ has characteristic $p$. Deduce there are infinitely many prime numbers.