CLASSIFYING SPACES AND BUNDLES

(1) Prove that each of the following is a fibre bundle (in each case, fix a basepoint of the base space to your own liking) and determine whether it is trivial or not.
  (a) $\mathbb{C}^{n+1} - \{0\} \to \mathbb{C}\mathrm{P}^n$
  (b) The closed Möbius strip $M = [0,1] \times [0,1]/(0,x) \sim (1, 1-x)$ mapping to S^1 given by the first coordinate.
  (c) The universal covering map $S^n \to \mathbb{R}\mathrm{P}^n$ for $n \geq 2$.

(2) Prove that the map $V_k(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^n)$ from the Stiefel manifold of k frames in n-space to the Grassmannian of k planes in n-space is a fibre bundle. Complete the definition of the tautological vector bundle on $\mathrm{Gr}_k(\mathbb{C}^n)$ and prove it is a vector bundle.

(3) Give an example of a map $X \to B$ of Hausdorff spaces such that X has an $\mathbb{R}$ vector space structure over B, but X is not a vector bundle over B. By a vector space structure over B, we mean that there exist
  (a) An addition map: $+ : X \times_B X \to X$ over B.
  (b) A scalar multiplication map $\cdot : \mathbb{R} \times X \to X$ over B
  (c) A zero section map $0 : B \to X$ over B.

  such that the diagrams corresponding to the vector space axioms commute, or, equivalently in this case, so that the induced structure on any fibre is a vector space.

(4) Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

  (a) Show that if the outer square and the right-hand square are both pull-back squares, then so too is the left-hand square.
  (b) Give an example where the outer square and the left-hand square are both pull-backs but the right hand square is not.

(5) Suppose B is a simply connected space having the homology of S^n where $n \geq 2$. Suppose $F \to E \to B$ is a Serre fibre sequence and F is path connected. Deduce the existence of the Wang long exact sequence

$$\to \mathrm{H}^k(E;\mathbb{Z}) \to \mathrm{H}^k(F;\mathbb{Z}) \xrightarrow{\theta} \mathrm{H}^{k-n+1}(F;\mathbb{Z}) \to \mathrm{H}^{k+1}(E;\mathbb{Z}) \to$$

(6) Let $\mathrm{Sp}(n)$ denote the compact symplectic group of $2n \times 2n$ unitary matrices A satisfying $A^T\Omega = \Omega A^{-1}$ where $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
  (a) Show that $\mathrm{Sp}(1)$ is homeomorphic to S^3. It may be helpful to view $\mathrm{Sp}(n)$ as the "orthogonal group" for the standard quaternions $\mathbb{H}$.
  (b) Calculate $\mathrm{H}^*(\mathrm{Sp}(n);\mathbb{Z})$, including the ring structure.

(7) Let $p : V_2(\mathbb{R}^n) \to V_1(\mathbb{R}^n)$ denote the map of Stiefel manifolds forgetting the second column: $[\vec{v}_1, \vec{v}_2] \mapsto \vec{v}_1$. Determine, with proof, the values of n for which p has a section.

(8) For all n, calculate the ring $\mathrm{H}^*(K(\mathbb{Z}, n), \mathbb{Q})$. Hence, or otherwise, calculate $\pi_n(S^m) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all values of n, $m \geq 2$. The cases $m = 0, 1$ are trivial.