CLASSIFYING SPACES AND BUNDLES

(1) Prove that each of the following is a fibre bundle (in each case, fix a basepoint of the base space to your own liking) and determine whether it is trivial or not.
   (a) $\mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$
   (b) The closed Möbius strip $M = [0, 1] \times [0, 1]/(0, x) \sim (1, 1-x)$ mapping to $S^1$ given by the first coordinate.
   (c) The universal covering map $S^n \to \mathbb{R}P^n$ for $n \geq 2$.

(2) Prove that the map $V_k(\mathbb{C}^n) \to \text{Gr}_k(\mathbb{C}^n)$ from the Stiefel manifold of $k$ frames in $n$-space to the Grassmannian of $k$ planes in $n$-space is a fibre bundle. Complete the definition of the tautological vector bundle on $\text{Gr}_k(\mathbb{C}^n)$ and prove it is a vector bundle.

(3) Give an example of a map $X \to B$ of Hausdorff spaces such that $X$ has an $\mathbb{R}$ vector space structure over $B$, but $X$ is not a vector bundle over $B$. By a vector space structure over $B$, we mean that there exist
   (a) An addition map: $+: X \times_B X \to X$ over $B$.
   (b) A scalar multiplication map $\cdot: \mathbb{R} \times X \to X$ over $B$
   (c) A zero section map $0: B \to X$ over $B$.
   such that the diagrams corresponding to the vector space axioms commute, or, equivalently in this case, so that the induced structure on any fibre is a vector space.

(4) Consider the following commutative diagram:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
C & \longrightarrow & Z
\end{array}
$$

(a) Show that if the outer square and the right-hand square are both pull-back squares, then so too is the left-hand square.
(b) Give an example where the outer square and the left-hand square are both pull-backs but the right-hand square is not.

(5) Suppose $B$ is a simply connected space having the homology of $S^n$ where $n \geq 2$. Suppose $F \to E \to B$ is a Serre fibre sequence and $F$ is path connected. Deduce the existence of the Wang long exact sequence

$$
\to H^k(E; \mathbb{Z}) \to H^k(F; \mathbb{Z}) \xrightarrow{\partial} H^{k-n+1}(F; \mathbb{Z}) \to H^{k+1}(E; \mathbb{Z}) \to
$$

(6) Let $\text{Sp}(n)$ denote the compact symplectic group of $2n \times 2n$ unitary matrices $A$ satisfying $A^T \Omega = \Omega A^{-1}$ where $\Omega = \left( \begin{array}{cc} 0 & I_n \\
-I_n & 0 \end{array} \right)$.
   (a) Show that $\text{Sp}(1)$ is homeomorphic to $S^3$. It may be helpful to view $\text{Sp}(n)$ as the “orthogonal group” for the standard quaternions $\mathbb{H}$.
   (b) Calculate $H^*(\text{Sp}(n); \mathbb{Z})$, including the ring structure.

(7) Let $p : V_2(\mathbb{R}^n) \to V_1(\mathbb{R}^n)$ denote the map of Stiefel manifolds forgetting the second column: $[\vec{v}_1, \vec{v}_2] \mapsto \vec{v}_1$. Determine, with proof, the values of $n$ for which $p$ has a section.
(8) For all $n$, calculate the ring $H^*(K(Z, n), \mathbb{Q})$. Hence, or otherwise, calculate $\pi_n(S^m) \otimes \mathbb{Z} \mathbb{Q}$ for all values of $n, m \geq 2$. The cases $m = 0, 1$ are trivial.