Chapter 1

Model Categories
1.1 Model Categories

The following definition \[\text{[Hov99]}\] is a modification of one due to Quillen \[\text{[Qui67]}\].

**Definition 1.1.1.** Let $\mathcal{C}$ be a category having all (small) limits and colimits. A *model structure* on $\mathcal{C}$ consists of three subcategories of $\mathcal{C}$ called *weak equivalences*, *cofibrations* and *fibrations* and two functorial ways of factoring maps $f$ in $\mathcal{C}$, either as $f = \alpha \circ \beta$ where $\alpha$ is a weak equivalence and a fibration and $\beta$ is a cofibration, or as $f = \gamma \circ \delta$ where $\gamma$ is a fibration and $\delta$ is a weak equivalence and a cofibration.

These data have to satisfy the following axioms:

1. If $f, g$ are morphisms in $\mathcal{C}$ such that $gf$ is defined, then if any two of $f, g, gf$ are weak equivalences, so is the third. (2-out-of-3 property)
2. If $f$ is a retract of $g$ and $g$ is a weak equivalence, cofibration or fibration, so is $f$.
3. If, in the solid commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i & & \downarrow p \\
B & \longrightarrow & Y
\end{array}
$$

the map $i$ is a cofibration and the map $p$ is a fibration, and at least one of these two maps is also a weak equivalence, then there exists a dotted arrow making the diagram commute.

**Notation 1.1.2.** A map that is a weak equivalence and a (co)fibration will be called a *trivial (co)fibration*. Weak equivalences will be written $A \sim B$, cofibrations will be written $A \rightarrow B$ and fibrations will be written $A \twoheadrightarrow B$.

A category equipped with a model structure will be called a *model category*.

**Notation 1.1.3.** In the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i & & \downarrow p \\
B & \longrightarrow & Y
\end{array}
$$

if the dotted arrow exists, we say that $A \rightarrow B$ has the *left lifting property* with respect to $X \rightarrow Y$ and that $X \rightarrow Y$ has the *right lifting property* with respect to $A \rightarrow B$.

**Remark 1.1.4.** The model structure axioms are dual with respect to fibrations and cofibrations, in that if $\mathcal{C}$ is a model category, then $\mathcal{C}^{\text{op}}$ is a model category as well, with the opposite of the category of cofibrations functioning as fibrations and vice versa. This duality manifests frequently in the theory, in that it is often sufficient to prove something for cofibrations and then say the case of fibrations is dual.

**Remark 1.1.5.** The fibrations are exactly the maps having the right lifting property with respect to trivial cofibrations, and the trivial cofibrations are exactly the maps having the right lifting property with respect to fibrations. Therefore, in a given model category, the weak equivalences and cofibrations determine the fibrations. Dually, the weak equivalences and fibrations determine the cofibrations.

**Remark 1.1.6.** A pushout of a (trivial) cofibration is a (trivial) cofibration, since we can detect (trivial) cofibrations by means of a left lifting property. The dual statement for fibrations and pullbacks also holds.
Example 1.1.7. Let $\textbf{Top}$ denote the category of topological spaces and continuous maps. We say a map $f : X \to Y$ in this category is a weak equivalence if

$$f_* : \pi_n(X, x) \to \pi_n(Y, f(y))$$

is an isomorphism (of pointed sets, groups or abelian groups) for all $x \in X$ and all $n \in \{0, 1, \ldots\}$.

Let $J$ denote the set of all inclusions $D^n \to D^n \times I$ sending $x$ to $(x, 0)$ for all $n$. We say $f : X \to Y$ is a Serre fibration if it has the left lifting property with respect to all maps in $J$.

There are two ways of defining Serre cofibrations. In the first place, we can define a Serre cofibration as a map $A \to B$ having the left lifting property with respect to all maps having the right lifting property with respect to the inclusions $\partial D^n \to D^n$ (yes, really).

Alternatively, we can define a relative cell complex to be a map $f : B \to A$ such that there is a sequence $B = B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$ where $B_{i+1}$ is obtained from $B_i$ by attaching cells and such that $A = \text{colim} B_i$. A map is a Serre cofibration if it is a retract of a relative cell complex. It is not proved here that these two definitions are equivalent, you can look at [Hov99, Chapter 2].

This gives us a model structure on $\textbf{Top}$, called the Quillen or classical model structure.

The main purpose of a model structure is to allow one to form and work with an associated homotopy category.

Definition 1.1.8. Let $C$ be a model category. Then the homotopy category, $\text{Ho} C$, of $C$ is the category with the same objects as $C$ and with morphisms obtained by adjoining (formal) inverses to weak equivalences. The definition is made more fully and precisely as [Hov99, Definition 1.2.1].

Remark 1.1.9. The first problem with $\text{Ho} C$ is that we do not know that $\text{Ho} C(X, Y)$ is a set. The second, and related problem, is that it is hard to calculate $\text{Ho} C(X, Y)$. The role of the fibrations and cofibrations in the model structure is to solve these two problems.

Definition 1.1.10. Write $\emptyset$ for the colimit of the empty diagram and $\text{pt}$ for the limit.

An object $X$ of a model category $C$ is cofibrant if the unique map $\emptyset \to X$ is a cofibration. It is fibrant if $X \to \text{pt}$ is a fibration.

Example 1.1.11. In the ordinary model structure on $\textbf{Top}$, the retracts of CW complexes are precisely the cofibrant objects. All objects are fibrant.

Example 1.1.12. We can factor any map $\emptyset \to X$ as $\emptyset \to QX \to X$. Then $QX$ is cofibrant and weakly equivalent to $X$. Any cofibrant object weakly equivalent to $X$ is called a cofibrant replacement. The dual concept is of fibrant replacement $X \to RX \to \text{pt}$. Observe that the fibrant replacement of a cofibrant object is cofibrant and vice versa. This means we can write down cofibrant–fibrant replacements.

In the standard model structure, we have no need for fibrant replacements, but we know cofibrant replacements as “CW approximations”.

Proposition 1.1.13 (Ken Brown’s Lemma). Suppose $C$ is a model category and $D$ is a category with weak equivalences (satisfying 2-out-of-3). Suppose $F : C \to D$ is a functor taking trivial cofibrations between cofibrant objects to weak equivalences, then $F$ takes all weak equivalences between cofibrant objects to weak equivalences.

The dual statement for fibrations and fibrant objects also holds.

In fact, as [Hov99] uses this lemma, it is necessary only for $C$ to satisfy the axioms of a model category and to contain finite coproducts (or finite products in the fibrant case).
Remark 1.1.17. "homotopy" is a cylinder object in this model structure, and a left homotopy is just what we would ordinarily call a right homotopy. This example of the sort of argument that model categories require. Suppose we have three maps \( f : X \to Y \), with evaluation at 0 and 1 being the maps to the diagonal \( X \) copies of \( X \). Among the more technical parts of this statement is the assertion that left-homotopy is a transitive relation on maps \( X \to Y \). Suppose \( X \) is cofibrant and \( Y \) is fibrant then the relations of left- and right-homotopy be transitive. For instance, if \( X \) is a topological space, we can form \( PX \) as \( X^I \) in the compact-open topology, with evaluation at 0 and 1 being the maps to \( X \). Using the adjunction between maps \( Y \to X^I \) and \( Y \times I \to X \), we see that here again we have recovered the usual notion of homotopy, but in a slightly harder-to-visualize way.

Definition 1.1.15. Let \( X \) be an object in a model category. A cylinder object on \( X \) is a factorization of the fold map \( X \amalg X \to X \) as 
\[
X \amalg X \to \text{Cyl}X \to X. 
\]
Note that we do not require this to be the specific functorial factorization of the model structure, although it certainly provides a cylinder object. A left homotopy between two morphisms \( f, g : X \to Y \) is a map \( H : \text{Cyl}X \to Y \) that yields \( f, g \) when composed with the two inclusions \( X \to X \amalg X \). Therefore the definition of \( \text{Cyl}X \) is a cylinder object in this model structure, and a left homotopy is just what we would ordinarily call a "homotopy".

Example 1.1.16. In the standard model structure on spaces, the inclusion \( X \amalg X \to X \times I \), including the copies of \( X \) at either end, is a cofibration (it's a relative cell complex). Therefore the definition of \( \text{Cyl}X \) is a cylinder object in this model structure, and a left homotopy is just what we would ordinarily call a “homotopy”.

Remark 1.1.17. Let \( X \) be an object in a model category. A path object, \( PX \), on \( X \) is a factorization of the diagonal \( X \to X \times X \) as 
\[
X \approx PX \to X \times X. 
\]
A right homotopy between \( f : Y \to X \) and \( g : Y \to X \) is a map \( H : Y \to PX \) that composes to give \( f \) or \( g \) in the analogous way. For instance, if \( X \) is a topological space, we can form \( PX \) as \( X^I \) in the compact-open topology, with evaluation at 0 and 1 being the maps to \( X \). Using the adjunction between maps \( Y \to X^I \) and \( Y \times I \to X \), we see that here again we have recovered the usual notion of homotopy, but in a slightly harder-to-visualize way.

Proposition 1.1.18. If \( X \) is cofibrant and \( Y \) is fibrant then the relations of left- and right-homotopy between maps \( X \to Y \) agree. Moreover, this relation is an equivalence relation.

Proof of a selected part of this statement. Among the more technical parts of this statement is the assertion that left-homotopy is a transitive relation on maps \( X \to Y \) when \( X \) is cofibrant. Let's do this, as an example of the sort of argument that model categories require. Suppose we have three maps \( f_1, f_2 \) and \( f_3 : X \to Y \), and left homotopies \( H_1 : X' \to Y \) and \( H_2 : X'' \to Y \) where \( X' \) and \( X'' \) are cylinders for \( X \).

Form \( Z \) as the pushout
\[
\begin{array}{ccc}
X & \xrightarrow{i_1} & X' \\
\downarrow{i''} & & \downarrow{}
\end{array}
\]

\[
\begin{array}{ccc}
X'' & \xrightarrow{i_3} & Z
\end{array}
\]

Proof. This proof is slightly tricky.

Suppose \( f : X \to Y \) is a weak equivalence of cofibrant objects. Factor \( X \amalg Y \to Y \) into a cofibration \( q : X \amalg Y \to Z \) followed by a trivial fibration \( p : Z \to Y \). Each of the two inclusion maps \( i_1 : X \to X \amalg Y \) and \( i_2 : Y \to X \amalg Y \) is a cofibration. Each of the two composite maps \( X, Y \to Z \) is a weak equivalence (2-out-of-3) and a cofibration. Both \( F(q \circ i_1) \) and \( F(q \circ i_2) \) are weak equivalences, as is \( F(p \circ q \circ i_2) = F(id_Y) \), so that \( F(p) \) is a weak equivalence, and so too is \( F(f) = F(p \circ q \circ i_1) \).

Notation 1.1.14. Let \( C_{cf} \) denote the full subcategory of \( C \) consisting of cofibrant–fibrant objects. This may not be a model category, since it may not be complete or cocomplete. Nonetheless, we can talk about \( Ho_{C_{cf}} \), the category obtained from \( C_{cf} \) by formally inverting weak equivalences, and Ken Brown’s lemma applies to \( C_{cf} \). It is not difficult to see ([Hov99, Prop 1.2.3]) that the obvious functor \( Ho_{C_{cf}} \to Ho_C \) is an equivalence of categories.

1.1.1 Homotopy

Definition 1.1.15. Let \( X \) be an object in a model category. A cylinder object on \( X \) is a factorization of the fold map \( X \amalg X \to X \) as 
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\]
Note that we do not require this to be the specific functorial factorization of the model structure, although it certainly provides a cylinder object. A left homotopy between two morphisms \( f, g : X \to Y \) is a map \( H : \text{Cyl}X \to Y \) that yields \( f, g \) when composed with the two inclusions \( X \to X \amalg X \). Therefore the definition of \( \text{Cyl}X \) is a cylinder object in this model structure, and a left homotopy is just what we would ordinarily call a “homotopy”.

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\[
X \approx PX \to X \times X. 
\]
A right homotopy between \( f : Y \to X \) and \( g : Y \to X \) is a map \( H : Y \to PX \) that composes to give \( f \) or \( g \) in the analogous way. For instance, if \( X \) is a topological space, we can form \( PX \) as \( X^I \) in the compact-open topology, with evaluation at 0 and 1 being the maps to \( X \). Using the adjunction between maps \( Y \to X^I \) and \( Y \times I \to X \), we see that here again we have recovered the usual notion of homotopy, but in a slightly harder-to-visualize way.

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\[
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\]

\[
\begin{array}{ccc}
X'' & \xrightarrow{i_3} & Z
\end{array}
\]
and we get a factorization of the fold map $X \coprod X \xrightarrow{j_0+j_1} Z \xrightarrow{f} X$. We remark that if $X$ is cofibrant, then $X \to X'$ (or $X \to X''$) is a cofibration, being the pushout of a cofibration. It is also a weak equivalence, due to the 2-out-of-3 property. Therefore, since $X$ is cofibrant, the map $X'' \to Z$ is a trivial cofibration.

Unfortunately, the map $j_0+j_1$ may not be a cofibration, although $t$ is necessarily a weak equivalence, since $X'' \to Z$ is a trivial cofibration and $Z \to X$ fits in a factorization of the identity $X \xrightarrow{\sim} X'' \to Z \to X$ so that 2-out-of-3 implies $Z \to X$ is a weak equivalence. Now take $X \coprod X \to Z$ and factor it as a cofibration followed by a trivial fibration: $X \coprod X \to Z' \xrightarrow{\sim} Z$. The object $Z'$ is a cylinder object for $X$ and it supports a left homotopy from $f_1$ to $f_3$.

**Proposition 1.1.19.** There is a well-defined functor $C_{cf} \to C_{cf}/\sim$ sending objects to objects and sending a map $f : X \to Y$ to the homotopy class of $f$.

The proof is not given here. See [Hov99, Section 1.2].

**Proposition 1.1.20.** Let $Y$ be a cofibrant–fibrant object and let $C/\sim (Y, \cdot) : C_{cf} \to \text{Set}$ denote the functor taking an object $X$ to the set of homotopy classes of maps $C(X,Y)/\sim$. Then $C/\sim (Y, \cdot)$ sends weak equivalences to bijections.

**Proof.** By Ken Brown's lemma (using isomorphisms of sets as weak equivalences in that category), it's sufficient to prove that it sends trivial fibrations to isomorphisms. That is, if $X \to Z$ is a trivial fibration, we want to show that $C(Y,X)/\sim \to C(Y,Z)/\sim$ is a bijection. Since $X \to Z$ is a trivial fibration, we can lift any map $Y \to Z$ to a map $Y \to Z$. This establishes surjectivity.

To show injectivity, suppose we have two maps $f, g : Y \to X$ that become (left) homotopic when we compose with $X \to Z$. Choose a left homotopy $H : Y' \to Z$ between them and consider

\[
\begin{array}{ccc}
Y & \xrightarrow{f+g} & X \\
\downarrow & & \downarrow \\
Y' & \to & Z
\end{array}
\]

the obvious lift gives us a homotopy between $f$ and $g$. \(\square\)

**Proposition 1.1.21.** A map $f : X \to Y$ between cofibrant–fibrant objects is a homotopy equivalence if and only if it is a weak equivalence.

**Proof.** The previous result says that if $f : X \to Y$ is a weak equivalence, then the induced map $C(Y,Y)/\sim \to C(Y,X)/\sim$ is a bijection. The class of the identity map then maps to a homotopy class of maps $g : Y \to X$, any one of which is a homotopy inverse for $f$ —admittedly we are skipping many details here.

The converse statement that a homotopy equivalence is a weak equivalence is surprisingly intricate. First observe that a map that is left homotopy equivalent to a weak equivalence is a weak equivalence. This follows from the diagram

\[
\begin{array}{ccc}
A & \xleftarrow{\sim} & B \\
\downarrow & \ & \downarrow \\
\text{Cyl } A & \to & B
\end{array}
\]
and the 2-out-of-3 property.

Now suppose \( f : X \to Y \) is a homotopy equivalence between cofibrant–fibrant objects. We can factor this as \( g : X \to Z \) followed by \( p : Z \to Y \). It suffices to show that \( p \) is a weak equivalence. Note that \( Z \) is cofibrant–fibrant and that \( g \) is therefore known to be a homotopy equivalence. Let \( f' \) and \( g' \) be homotopy inverses for \( f \) and \( g \), and let \( H : Y' \to Y \) be a left homotopy from \( f f' \) to \( \text{id}_Y \). Define \( H' \) as a lift

\[
\begin{array}{ccc}
Y' & \to & Z \\
\downarrow q & & \downarrow p \\
Y & & \\
\end{array}
\]

and let \( q = H'_1 : Y \to Z \). The map \( q \) has been produced specifically so that \( pq = \text{id}_Y \), and so that \( H' \) is a left homotopy from \( g f' \) to \( q \). Then \( q p \sim g f' p \sim g f' p g' = g f' f g' \sim \text{id}_Z \).

In particular, we have reduced the problem to showing that \( p \) is a weak equivalence when \( p : Z \to Y \) is the retraction map of a deformation retract. This is relatively easy:

\[
\begin{array}{ccc}
Z & \longrightarrow & Z \\
\downarrow p & & \downarrow p \\
Y & \longrightarrow & \\
\end{array}
\]

and the middle map is a weak equivalence, so the outer map is as well (weak equivalences being closed under retracts).

**Corollary 1.1.22** (Whitehead’s theorem). If \( X \to Y \) is a weak equivalence of CW complexes, then it is a homotopy equivalence.

Now we come to the point of the whole discussion:

**Corollary 1.1.23.** The equivalent categories \( \text{Ho} \mathcal{C}_{cf} = \text{Ho} \mathcal{C} \) are equivalent to the category \( \mathcal{C}_{cf}/\sim \).

This means we have a method of calculating \( \text{Ho}(X, Y) \): namely, replace \( X \) and \( Y \) by weakly equivalent cofibrant-fibrant objects \( X' \) and \( Y' \), then calculate \( \mathcal{C}(X', Y')/\sim \).

In fact, you can do a bit better

**Corollary 1.1.24.** Let \( X' \to X \) be a cofibrant replacement of \( X \) in \( \mathcal{C} \) and \( Y \to Y' \) a fibrant replacement. Then \( \text{Ho}(X, Y) = \mathcal{C}(X', Y')/\sim \).

This follows from [Hov99, Proposition 1.2.5] and the previous corollary.

**Example 1.1.25.** There is another model structure on \( \text{Top} \), the Strom model structure, established in [Str72], in which the role of the weak equivalences is played by the homotopy equivalences of spaces, the fibrations are the maps satisfying the right lifting property with respect to the inclusions \( X \to X \times I \) for all spaces \( X \) and the closed topological cofibrations.

**Example 1.1.26.** This example is [Hov99, Section 2.3]. Let \( R \) be a ring, and work in the category of left \( R \)-modules. We assume you know what a complex of \( R \)-modules is (with homological grading), and what a map of complexes looks like. A map \( f : A_* \to B_* \) is a quasi-isomorphism if \( f_* : H_i(A_*) \to H_i(B_*) \) is an isomorphism for all \( i \).
For any integer $n$, let $S^n$ denote the complex that has $R$ in the $n$th position and 0 elsewhere. Let $D^{n+1}$ denote the complex that has $R$ in the $n$-th and the $n + 1$st position, and where the nontrivial differential is an identity. There is an inclusion map of complexes $i_n : S^n \rightarrow D^{n+1}$.

Call a map of complexes a **fibration** if it has the right lifting property with respect to all maps $0 \rightarrow D^n$, and a **cofibration** if it has the left lifting property with respect to all maps having the right lifting property with respect to the inclusions $i_n$.

This produces a model structure on the category of chain complexes of $R$-modules. Some claims, all of which are proved in [Hov99]:

1. The fibrations are precisely the levelwise surjective maps.
2. A map is a trivial fibration if and only if it has the right lifting property w.r.t. the $i_n$.
3. A bounded-below chain complex of projective modules is cofibrant. Any cofibrant object is levelwise projective, but there exist unbounded, levelwise projective and non-cofibrant objects.

### 1.2 Quillen adjunctions

According to the experts, the “right” notion of a morphism between model categories is the following.

**Definition 1.2.1.** Suppose $C$ and $D$ are model categories (i.e., categories and model structures). Then a functor $F : C \rightarrow D$ is a **left Quillen functor** if it has a right adjoint $U$ and $F$ preserves cofibrations and trivial cofibrations. A functor $U : D \rightarrow C$ is a **right Quillen functor** if it is a right adjoint and preserves fibrations and trivial fibrations.

**Proposition 1.2.2.** Let $F \dashv U$ be an adjoint pair in which $F$ is left Quillen or $U$ is right Quillen. Then $F$ is left Quillen and $U$ is right Quillen.

**Remark 1.2.3.** Ken Brown’s lemma implies that a left Quillen functor preserves weak equivalences between cofibrant objects, and a right Quillen functor preserves weak equivalences between fibrant objects.

This means that we can form derived functors of Quillen functors.

**Construction 1.2.4.** Suppose $F : C \rightarrow D$ is a left Quillen functor. The **total left derived functor** $LF$ of $F$ is the functor

$$\text{Ho} C \rightarrow \text{Ho} D$$

that on objects $X \in C$ is $F(QX)$—$Q$ being the cofibrant replacement functor. The **total right derived functor** $RU$ of $U$ is defined dually.

**Remark 1.2.5.** We haven’t quite justified the assertion that this functor is defined, and we’re not going to do so explicitly. You can do it yourself.

**Proposition 1.2.6.** If $F \dashv U$ is a Quillen adjunction, with $F : C \rightarrow D$, then the pair of functors $QF : \text{Ho} C \Rightarrow \text{Ho} D : RU$ is an adjoint pair of functors.

**Proof.** We wish to establish a natural isomorphism of sets

$$\text{Ho} D(FQX, Y) \xrightarrow{\cong} \text{Ho} C(X, URY).$$
It is enough to establish a natural isomorphism

\[ D(FQX, RY) / \sim \rightarrow C(QXURY) / \sim \]

which looks a lot like the adjunction we already had between \( F \) and \( U \), applied to the objects \( QX \) and \( RY \). The only question is whether this adjunction isomorphism (for \( C \), \( D \)) preserves the relation “is homotopic to”. Call the adjunction isomorphism \( \phi \).

Suppose \( f, g : FX \rightarrow Y \) are two maps in \( D \), where \( X \) is cofibrant and \( Y \) fibrant. Let’s show that if \( \phi f \) is homotopic to \( \phi g \), then \( f \) is homotopic to \( g \). The other argument is dual (using right instead of left homotopy). So suppose \( X' \) is a cylinder object for \( X \) and \( H \) a left homotopy \( H : X' \rightarrow UY \) between \( \phi f \) and \( \phi g \). Then \( FX' \) is a cylinder object for \( FX \), since \( F \) preserves coproducts, cofibrations and trivial cofibrations, and \( \phi^{-1} H \) is a left homotopy from \( f \) to \( g \).

**Definition 1.2.7.** A Quillen adjunction \( F \dashv U \) is a Quillen equivalence if the following holds: For all cofibrant \( X \in C \) and all fibrant \( Y \in D \), the map \( f : FX \rightarrow Y \) is a weak equivalence if and only if the adjoint map \( X \rightarrow UY \) is a weak equivalence.

**Remark 1.2.8.** The derived functors of a Quillen equivalence are equivalences of categories.
Chapter 2

Simplicial Sets

2.1 Definition

Let $\Delta$ denote the simplicial category where the objects are

$$[n] = \{0, 1, \ldots, n\}$$

for $n \geq 0$ and the maps $\Delta([n], [k])$ are the set of weakly order-preserving maps of sets.

**Lemma 2.1.1.** Every map $f$ in $\Delta$ can be factored uniquely as a surjection followed by an injection.

**Definition 2.1.2.** Let $d^i : [n-1] \to [n]$ denote the injective map $[n-1] \to [n]$ skipping $i \in \{0, 1, \ldots, n\}$. This is called the $i$-th (standard) coface map. Let $s^i : [n] \to [n-1]$ be the surjective map identifying $i$ and $i+1$, the $i$-th (standard) codegeneracy map.

**Lemma 2.1.3.** Every injective map in $\Delta$ can be factored as a composite of coface maps, and every surjective map can be factored as a composite of codegeneracy maps. Consequently, every map in $\Delta$ is a composite of coface and codegeneracy maps.

**Lemma 2.1.4.** The following relations all hold

$$
\begin{align*}
\quad & d^i d^j = d^i d^{j-1} & i < j \\
\quad & s^i d^j = d^i s^{j-1} & i < j \\
\quad & s^i d^j = \text{id} & i = j, i = j + 1 \\
\quad & s^i d^j = d^{i-1} s^j & i > j + 1 \\
\quad & s^i s^j = s^{i-1} s^j & i > j 
\end{align*}
$$

and suffice to generate all relations in the simplicial category.

**Definition 2.1.5.** Let $C$ be a category. A simplicial object in $C$ is a functor $X_* : \Delta^{op} \to C$. The notation $X_n$ is used for $X_*([n])$. The images $X_*(d^i) : X_{n+1} \to X_n$ are written $d_i$ and are called face maps. Similarly, $X_*(s^i) = s_i$ are called degeneracy maps. These are subject to the simplicial identities which are the duals of the relations of Lemma 2.1.4.
A map of simplicial objects is a natural transformation of functors, or equivalently, a sequence of levelwise maps \( X_n \to Y_n \) compatible with the simplicial maps. The category of simplicial objects in \( C \) will be written \( sC \).

**Lemma 2.1.6.** If \( C \) is a category with limits and colimits, then so is \( sC \); limits and colimits being formed objectwise.

**Example 2.1.7.** The category of simplicial sets, \( sSet \) is particularly important. If \( X_* \) is a simplicial set, then we can distinguish two kinds of element in \( X_n \): those that are in the image of some degeneracy map the *degenerate elements*, and those that are not.

Let us define some standard objects in this category. In the first place, there is \( \Delta[n] \) which sends \([k]\) to \( \Delta([k],[n]) \). This is a the (simplicial) \( n \)-simplex. There is also \( d\Delta[n] \), which is the subobject of \( \Delta[n] \) consisting of all those maps \([k] \to [n]\) that are not surjective.

For each \( r \), there is \( \Lambda^r[n] \) that can be constructed as follows: Let \( \mathcal{D} \) be the category where the objects are non-identity injective maps \([k] \to [n]\) where the image contains \( r \). Each map yields a map \( \Delta[k] \to \Delta[n] \) of simplicial sets, then \( \Lambda^r[n] \) is the colimit over \( \mathcal{D} \) of the \( \Delta[k] \).

When we draw a picture of a simplicial set, we usually draw the non-degenerate simplices only. There are several reasons why degenerate simplices are included in the structure. For instance, \( \Delta[1] \times \Delta[1] \) has two nondegenerate 2-simplices that arise from the degenerate simplices of \( \Delta[1] \).

**Definition 2.1.8.** A cosimplicial object in \( C \) is a functor \( X^* : \Delta \to C \). A cosimplicial object is a sequence of objects in \( C \) equipped with coface and codegeneracy maps satisfying the relations of Lemma 2.1.4.

**Example 2.1.9.** There is a standard cosimplicial topological space. This is given on objects by

\[
\Delta^n_i = \left\{ x \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1, \quad x_i \geq 0, \quad \forall i \right\}.
\]

The \( i \)-th coface map \( \Delta^{n-1}_i \to \Delta^n_i \) is given by including \((x_1, \ldots, x_n) \to (x_1, \ldots, 0, \ldots, x_n)\), inserting a 0 in the \( i \)-th position, and the \( i \)-th codegeneracy map is given by the map \( \Delta^n_i \to \Delta^{n-1}_i \) sending \( x \to (x_0, x_1, \ldots, x_i + x_{i+1}, \ldots, x_n) \).

**Construction 2.1.10.** Let \( C \) be a category with all colimits and let \( A^* \) be a cosimplicial object in \( C \). We can construct a functor

\[
| \cdot |_A : sSet \to C \quad \text{(realization)}
\]

in the following way:

Suppose \( X_* \) is a simplicial set. Let \( \Delta X \) denote the category where the objects are maps \( \Delta[n] \to X \) and the maps are commuting triangles. This is called the category of simplices in \( X \). Tautologically, \( \operatorname{colim}_{(\Delta[n] \to X) \in \Delta X} \Delta[n] = X \). The \( A \)-realization functor is then

\[
|X| = \operatorname{colim}_{(\Delta[n] \to X) \in \Delta X} A^n.
\]

In the case where \( C = \text{Top} \) (or similar) and \( A = \Delta^*_r \), this construction is called the geometric realization of the simplicial set \( X_* \), and is written \( |X_*| \).

**Remark 2.1.11.** In order to produce \(|X_*|\), it is sufficient to consider the nondegenerate simplices of \( X_* \).
Construction 2.1.12. Let $C$ be a category with all colimits and let $A^*$ be a cosimplicial object in $sC$. We can construct a functor $\text{Sing}^A : C \to s\text{Set}$ by setting $\text{Sing}^A(Y)_n = C(A^n, X)$. The cosimplicial structure maps in $A^*$ then immediately yield the simplicial structure maps in $\text{Sing}^A(Y)$.

When $A = \Delta^n$, this is a familiar construction: $\text{Sing}^A(Y)_n = \text{Top}(\Delta^n, Y)$ is the collection of singular $n$-simplices in $Y$. This is the basis for the free abelian group $C_n^{\text{sing}}(Y)$ used to calculate singular homology.

Proposition 2.1.13. Let $C$ be a category having all colimits. Then the two constructions above form an adjoint pair of functors $|\cdot| : A^{\text{co}} \to \text{Sing}^A$.

Proof.

$$C(\text{colim}_X A^n, Y) = \text{lim}_X C(A^n, Y) = \text{lim}_X \text{Sing}^A(Y)_n = \text{lim}_X s\text{Set}(\Delta[n], \text{Sing}^A(Y)_n) = s\text{Set}(X, \text{Sing}^A(Y)_n)$$

$\square$

Definition 2.1.14. Let $K$ denote the category of compactly generated spaces, also known as “Kelly spaces”. The category $K$ is a subcategory of $\text{Top}$ and the inclusion $K \to \text{Top}$ is left-adjoint to a “Kellification”, functor so that $K$ is closed under all colimits in $\text{Top}$. It is notably not closed under ordinary products of spaces, instead there is a product in $K$ given by replacing $X \times Y$ by its Kellification—this has the same underlying set, but possibly different closed subsets.

Lemma 2.1.15. The geometric realization functor $|\cdot| : s\text{Set} \to K$ preserves finite products.

Outline of proof. That is, we assert $|X_1 \times X_2| \approx |X_1| \times |X_2|$. Observe that there is a map in the forward direction here, and it suffices to prove it is a homeomorphism.

It is a feature of both the category of sets and of $K$ that there is a functor $\cdot \times A$ that is a left adjoint—see the Appendix to Gaunce Lewis’ thesis for a proof of this for $K$. In particular, finite products commute with all colimits in both categories.

Moreover, $|\cdot|$ is a left adjoint, and commutes with all colimits. We consequently have a reduction

$$|X_1 \times X_2| = |\text{colim}_{\Delta X_1} \Delta[n] \times X_2| = \text{colim}_{\Delta X_1} |\Delta[n] \times X_2| = \text{colim}_{\Delta X_1, \Delta X_2} |\Delta[n]| \times |\Delta[m]|$$

and

$$|X_1| \times |X_2| = \text{colim}_{\Delta X_1, \Delta X_2} |\Delta[n]| \times |\Delta[m]|$$

so it suffices to prove that the natural map

$$\nu : |\Delta[n]| \times |\Delta[m]| \to |\Delta[n]| \times |\Delta[m]|$$

is a homeomorphism. The target here is clearly compact.

One sees that the nondegenerate simplices of $\Delta[n] \times \Delta[m]$ correspond to totally ordered subsets of the partially-ordered set $|n| \times |m|$. There are only finitely many of these, so $|\Delta[n| \times |\Delta[m]|$ is compact. Therefore it suffices to show $\nu$ is bijective.

It then suffices to show that every point in $|\Delta[n]| \times |\Delta[m]|$ arises from realizing one of the maximal (i.e., $n + m$-dimensional) nondegenerate simplices in $\Delta[n] \times \Delta[m]$. This is done in the last part of [Hov99, Lemma 3.1.8].

$\square$
2.2 The model structure on simplicial sets

2.2.1 Summary

Definition 2.2.1. A map \( f : X \to Y \) is a \textit{weak equivalence of simplicial sets} if \(|f|\) is a weak equivalence of topological spaces. Since \(|X|\) and \(|Y|\) are CW complexes, this is the same as being a homotopy equivalence of spaces.

Example 2.2.2. Note that the three spaces \( |\Lambda^2_i| \) are all homeomorphic as spaces, but the \( \Lambda^2_i \) are pairwise non-isomorphic as simplicial sets.

Definition 2.2.3. A map \( f : X \to Y \) is a \textit{cofibration of simplicial sets} if it is levelwise injective.

Definition 2.2.4. A map \( f : X \to Y \) is a \textit{fibration of simplicial sets} or a \textit{Kan fibration} if it has the following right lifting property

\[
\begin{array}{ccc}
\Lambda^i[n] & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\Delta[n] & \longrightarrow & Y
\end{array}
\] (2.1)

for all \( n \) and all \( i \in \{0, \ldots, n\} \).

Notation 2.2.5. A simplicial set \( K \) such that \( K \to \Delta[0] \) is a fibration (i.e., a fibrant object in this structure) is called a \textit{Kan complex}.

Remark 2.2.6. A simplicial set \( X \), such that \( X \to \text{pt} \) satisfies the lifting property in (2.1) for all \( n \) and all \( i \in \{1, \ldots, n-1\} \) is called a \textit{quasicategory}. We may return to this definition later.

Theorem 2.2.7. The weak equivalences, cofibrations and fibrations defined above form a model structure.

2.2.2 Cofibrant generation

We won’t prove this in full detail, since we’ve allotted no more than two lectures to it. Here are the main ideas. The full proof can be assembled from [Hov99, Section 2.1] and [Hov99, Section 3.2-3.6].

Notation 2.2.8. Let \( I \) be a collection of maps in a category \( C \) having all colimits. The notation \( I - \text{inj} \) denotes the collection of maps that have the r.l.p. with respect to the maps in \( I \). Let \( I - \text{cof} \) denote the set of maps having the l.l.p. w.r.t. \( I - \text{inj} \). Let \( I - \text{cell} \) (the relative \( I \) cell complexes) denote the smallest collection of maps that is

1. closed under direct colimits and
2. closed under coproducts (this actually follows from the other two axioms)
3. contains all pushouts of maps in \( I \).

Lemma 2.2.9. Any retract of a map in \( I - \text{cof} \) is in \( I - \text{cof} \).

Lemma 2.2.10. Any map in \( I - \text{cell} \) is in \( I - \text{cof} \).
**Definition 2.2.11.** Let $I$ be a collection of morphisms in a category $C$. An object $X \in C$ is *small* (relative to $I$) if, for all direct systems $Y_1 \to Y_2 \to \ldots$ of maps in $I$, the map
\[
\operatorname{colim}_i C(X, Y_i) \to C(X, \operatorname{colim}_i Y_i)
\]
is a bijection.

**Remark 2.2.12.** The small objects in the category of sets are the finite sets. The small objects in $\mathbf{sSet}$ are the simplicial sets having finitely many nondegenerate simplices.

**Remark 2.2.13.** There is a generalization of smallness to $\kappa$-smallness, where the direct systems are indexed over other ordinals than $\omega$.

**Theorem 2.2.14 (The small object argument, finite version).** Let $C$ be a category having all colimits, and that $I$ is a set of maps. Suppose the domains of the maps in $I$ are small relative to $I$–cell. There is a functorial factorization of all maps $f$ in $C$ into $\delta(f) \circ \gamma(f)$ where $\delta(f)$ is in $I$–inj and $\gamma(f)$ is in $I$–cell (and in particular, in $I$–cof).

There is also a version of this for more general notions of smallness.

**General idea of proof.** Let $f : X \to Y$ be a map in $C$. We want to produce a factorization $X \to X' \to Y$ where $X \to X'$ is in $I$–cell and $X' \to Y$ has the r.l.p. w.r.t. $I$. To what extent does $X \to Y$ fail to have that lifting property already? Suppose there is a diagram
\[
\begin{array}{ccc}
A & \to & X \\
\downarrow & & \downarrow \\
B & \to & Y
\end{array}
\]
where $A \to B$ is in $I$. There may not be a lift along $X \to Y$, but we can replace $X$ by $X'$, the pushout of $B \leftarrow A \to X$. Then there is a factorization $X \to X' \to Y$ where the first map is $I$-cellular and the second map is closer to being in $I$–inj because at least in the diagram
\[
\begin{array}{ccc}
A & \to & X' \\
\downarrow & & \downarrow \\
B & \to & Y
\end{array}
\]
where $A$ is the composite $A \to X \to X'$, there is a lift.

The “small object argument” is an argument that says some (infinitely repeated) application of this idea does actually lead to a functorial factorization. □

**Lemma 2.2.15 (The Retract Argument).** Suppose $f = pi$ is a factorization of a map in a category where $f$ has the l.l.p. w.r.t. $p$. Then $f$ is a retract of $i$.

**Proof.** Write $i : A \to B$ and $p : B \to C$ and consider the lift in
\[
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{p} & C
\end{array}
\]
Then the diagram does the job.

Lemma 2.2.16. Assume I is small relative to I – cell. Any map in I – cof is a retract of a map in I – cell.

Proof of lemma. Let f : X → Y be a map in I – cof. Using the small object argument, we can factor it as a composite X → X’ → Y where the first map is in I – cell (and so in I – cof) and the second is in I – inj. Then use the retract argument.

Theorem 2.2.17 (Cofibrantly generated model structures). Let C be a category containing all limits and colimits. Let W be a subcategory of weak equivalences closed under retracts and satisfying the two-out-of-three property. Let I and J be two sets of map in C. Suppose the domains of the maps in I and J are each small relative to I – cell, J – cell respectively, and further that

1. J – cell ⊆ W ∩ I – cof
2. I – inj = W ∩ J – inj

Then there is a model structure on C where I – cof are the cofibrations, J – inj are the fibrations and I – inj are the trivial fibrations.

Remark 2.2.18. A model category admitting this sort of description is called cofibrantly generated.

Partial proof. For notational convenience, set I – cof to be the cofibrations and J – inj to be the fibrations. It is also easy to show that these are closed under composition. The hypotheses ensure that I – inj is exactly the trivial fibrations.

It is easy to verify that cofibrations and fibrations are closed under retracts—this is just a diagram chase in each case.

The small-object argument can be applied to give functorial factorizations of any map f into a map in J – cell followed by one in J – inj, and similarly for I. The first of these is the functorial → · → factorization. The second is the → · → factorization.

Finally we turn to the lifting axioms.

A cofibration has the l.l.p. w.r.t. I – inj, the trivial fibrations, by hypothesis. The case of a trivial cofibration is a little worse. A trivial cofibration f : A → B can be factored into a map h : A → A’ in J – cell, followed by a trivial fibration g : A’ → B. Since f has the lifting property w.r.t. the trivial fibration g, the retract lemma tells us that f is a retract of h, a map in J – cell. It follows that f is in J – cof, so has the lifting property against fibrations. It even follows that J – cof consists of precisely the trivial cofibrations.

2.2.3 Cofibrant generation of the structure on simplicial sets

Now we have to outline why the structure on simplicial sets fits into this structure.

Notation 2.2.19. Until further notice, I will denote the set of all canonical inclusions ∂Δ[n] → Δ[n], and J the set of all the canonical inclusions Λ[r][n] → Δ[n].
Proposition 2.2.20. The following are equivalent:

1. \( I \rightarrow \text{cof} \).
2. Relative \( I \)-cell complexes
3. Injective maps of simplicial sets

Proof. It is easy to verify that relative \( I \)-cell complexes are injective, and it is easy also to verify that retracts of injective maps are injective. Since every element of \( I \rightarrow \text{cof} \) is a retract of a relative cell complex, this implies that all the maps we are considering here are injective.

To finish the argument suffices to show that any injective map is actually a relative cell complex. This is routine enough, and is done in detail in [Hov99, 3.2.2]

We now try to apply Theorem 2.2.17.

Remark 2.2.21. For convenience, let us recapitulate the hypotheses of that theorem:

1. \( C \) contains all limits and colimits—done.
2. \( W \) is closed under retracts and satisfies 2-out-of-3—obvious.
3. Smallness—satisfied (the domains have only finitely many nondegenerate simplices in each case).
4. \( J \rightarrow \text{cell} \subseteq W \cap I \rightarrow \text{cof} \).
5. \( I \rightarrow \text{inj} = W \cap J \rightarrow \text{inj} \).

Proposition 2.2.22. Axiom 4 is satisfied.

In fact, more is true. The maps in \( J \rightarrow \text{cof} \) are called anodyne extensions, and they are all trivial cofibrations.

Proof. Since \( J \subseteq I \rightarrow \text{cof} \), it is formal that \( J \rightarrow \text{cof} \subseteq I \rightarrow \text{cof} \cap I \rightarrow \text{cof} = I \rightarrow \text{cof} \).

It remains to show that anodyne extensions are weak equivalences. It is formal from the \(| \cdot | \dashv \text{Sing} \) adjunction that the realization of an anodyne extension is in \(| J \rightarrow \text{cof} \), where \(| J \) denotes the geometric realization of the maps in \( J \). But the maps in \(| J \rightarrow \text{inj} \) are exactly the Serre fibrations, and so \(| J \rightarrow \text{cof} \) are the trivial Serre cofibrations, in particular, they are weak equivalences.

So only Axiom 5 remains. This would take a long time to prove, so we will not do it.

Proposition 2.2.23 (Hovey 3.2.6). If \( f \in I \rightarrow \text{inj} \), then \( f \) is a trivial fibration.

You can look in [Hov99] for a proof of this. Proving \( f \) is a fibration does not require anything that we haven’t done already, since we can produce the maps \( f \) as \( I \rightarrow \text{cell} \) complexes, against which \( f \) has the r.l.p. Proving that \( f \) is a homotopy equivalence is not difficult either, but it does involve the long exact sequence of a fibration (in classical homotopy theory), which we will discuss later.

Proposition 2.2.24. If \( f \) is a trivial fibration, then \( f \) is in \( I \rightarrow \text{inj} \).

Heavy combinatorics (i.e., the theory of minimal fibrations [Hov99, Definition 3.5.5] or [GJ99, Section I.10]) is used to reduce this to a statement about homotopy groups of simplicial sets. We will therefore devote some time to homotopy groups later, and you will have to trust me that the argument can be made in a not-circular way.

In the process of proving this, one proves most of the following theorem
Theorem 2.2.25. The realization and singular functors

\[ |\cdot| : \text{sSet} \rightleftarrows \text{K: Sing} \]

form a Quillen equivalence.

Proof. To see this is a Quillen adjunction, we concentrate on the right adjoint, Sing. Suppose \( f : X \to Y \) is a Serre fibration, and there is a diagram

\[
\begin{array}{ccc}
|\Lambda^1[n]| & \longrightarrow & X \\
\downarrow & & \downarrow \\
|\Delta[n]| & \longrightarrow & Y
\end{array}
\]

Since the left hand vertical map is a cellular inclusion and a weak equivalence, there is a lift in this diagram, and so, after applying the adjunction, we get a lift in

\[
\begin{array}{ccc}
\Lambda^1[n] & \longrightarrow & \text{Sing} X \\
\downarrow & & \downarrow \\
\Delta[n] & \longrightarrow & \text{Sing} Y
\end{array}
\]

The same argument also applies to \( \partial \Delta[n] \to \Delta[n] \) when \( f \) is a trivial Serre fibration. This handles the “Quillen adjunction” part of the theorem.

For the equivalence, let \( K \) be a simplicial set and \( X \) be a \( k \)-space, we have to show that \( |K| \to X \) is a weak equivalence if and only if \( K \to \text{Sing} X \) is a weak equivalence. This is means showing that \( |K| \to X \) is a weak equivalence if and only if \( |K| \to |\text{Sing} X| \) is a weak equivalence, which in turn is equivalent to showing that the natural map \( |\text{Sing} X| \to X \) is a weak equivalence. This is true, but we will postpone explaining why until after we describe homotopy groups.

Remark 2.2.26. As part of this Quillen equivalence, there is the statement that if \( f \) is a cofibration, then \( |f| \) is a cofibration. In fact, if \( f \) is a cofibration, then by 2.2.20, it is a relative cell complex (in what we called \( I- \) cell) and we can see directly that \( |f| \) is a relative cell complex of topological spaces.

In particular, \( \text{Ho sSet} \) is an equivalent category to \( \text{Ho K} \).

Two further facts about realizations of simplicial sets that we will not prove are given.

Theorem 2.2.27. The realization functor \( |\cdot| \) preserves all finite limits, i.e., limits of finite diagrams.

Sketch of proof. This appears as [Hov99, Lemma 3.2.4]. The idea is that all finite limits can be constructed by iterating finite products (for which we already know this result) and equalizers.

Definition 2.2.28. Let \( f, g : A \to B \) be two maps in a category. The equalizer of \( f, g \) is the limit of the diagram \( A \rightrightarrows B \).

To show that \( |\cdot| \) preserves equalizers, argue as follows. Let \( f, g : A \to B \) be two maps of simplicial sets, let \( K \) be the simplicial set equalizer. This is a subobject of \( A \). Let \( Z \) be the topological equalizer of \( |f|, |g| \). That is, this is the subspace of \( |A| \) on which the two cellular maps \( |f| \) and \( |g| \) agree. The space \( Z \) is a closed CW subspace of \( |A| \), and the functorial inclusion \( |K| \to |A| \) factors through \( Z \). It suffices to verify that the inclusion \( |K| \to Z \) is surjective, and this can be done on a simplex-by-simplex basis.

Theorem 2.2.29 (Quillen). If \( f : X \to Y \) is a Kan fibration of simplicial sets, then \( |f| \) is a Serre fibration.

This appears as [GJ99, Theorem 10.10].
Chapter 3

Homotopy theory of simplicial sets and spaces

3.1 Pointed model categories

**Notation 3.1.1.** Every model category has an initial object $\emptyset$ and a terminal object $\text{pt}$. A model category is said to be **pointed** if $\emptyset \rightarrow \text{pt}$ is an isomorphism. In particular, this implies that every object $X$ is equipped with a unique map $x_0 : \text{pt} \rightarrow X$ called the **basepoint** of $X$.

**Construction 3.1.2.** If $C$ is a model category, then we can form the associated pointed model category $C_+$ where the objects are pairs $(X, x_0)$, with $X \in C$ and $x_0 : \text{pt} \rightarrow X$ being a morphism (i.e., a choice of basepoint). Morphisms in $C_+$ are required to send basepoints to basepoints. Weak equivalences (resp. cofibrations, fibrations) are the maps that are weak equivalences (resp. cofibrations, fibrations) after forgetting the basepoint.

There is a functor $C \rightarrow C_+$ given by sending the object $X$ to $X \coprod \text{pt}$, pointed at the disjoint basepoint, and a forgetful functor $C_+ \rightarrow C$, forgetting basepoints. These functors form a Quillen adjunction.

**Remark 3.1.3.** Frequently, when working in a pointed category, we will write $X$ but mean $(X, x_0)$.

**Remark 3.1.4.** The Quillen equivalence of $sSet$ and $K$ extends to an equivalence of pointed categories.

3.2 Cartesian structure

**Definition 3.2.1.** Let $X$ and $Y$ be simplicial sets. Let $\text{Map}(X, Y)_*$ denote the simplicial set having $sSet(X \times \Delta^n, Y)$ as its $n$-th level. Since $\Delta^*$ form a cosimplicial set, this makes sense. The construction $\text{Map}(X, Y)_*$ is functorial in both variables (contravariantly in the first).

**Proposition 3.2.2.** For a fixed simplicial set $Y$, the functors $\cdot \times Y \dashv \text{Map}(Y, \cdot)$ form an adjoint pair.

**Proof.** Exercise. \qed

The following theorem sets a pattern for many similar theorems in homotopy theory, and is very useful.
**Theorem 3.2.3** (Mapping theorem). Suppose \( i : K \to L \) is a cofibration (injective map) and \( p : X \to Y \) is a fibration of simplicial sets, then the induced map
\[
\text{Map}(L, X) \to \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)
\]
is a fibration. It is a trivial fibration if either \( i \) or \( p \) is also a weak equivalence.

**Construction 3.2.4.** Note that \( \text{Map}(\cdot, pt) \) is a fibration. It is a trivial fibration if either \( i \) or \( p \) is also a weak equivalence.

**Remark 3.2.5.** The analogue of Theorem 3.2.3 holds in the pointed case.

**Remark 3.2.6.** An entirely analogous story can be told about the usual model structure on \( K \) (i.e., the restriction of the structure on \( \text{Top} \)). Here we have an internal mapping object \( \text{Map}(X, Y) = \mathcal{C}(X, Y) \), a pointed analogue, \( \text{Map}_+ (X, Y) \) and the smash product \( X \wedge Y \). The mapping theorem, 3.2.3, also applies in this case.

**Corollary 3.2.7** (Corollary of Theorem 3.2.3). Work in either the category of simplicial sets or the usual model structure on \( K \). Let \( L \) be a cofibrant object. Then the adjoint functors \( \cdot \times L \) and \( \text{Map}(L, \cdot) \) form a Quillen adjunction.

Similarly, in the pointed cases, \( \cdot \wedge L \) and \( \text{Map}_+ (L, \cdot) \) form a Quillen adjunction.

**Proof.** Apply the theorem with \( i : \emptyset \to L \) and \( p : X \to Y \) a (trivial) fibration. This suffices to show that \( \text{Map}(L, \cdot) \) preserves (trivial) fibrations, and so is a right Quillen functor. The pointed case is an exercise.

In particular, the adjunction descends to an adjunction on homotopy categories.

**Proposition 3.2.8.** Suppose \( i : L \to K \) is a weak equivalence, then the natural transformation \( \text{Map}(K, \cdot) \to \text{Map}(L, \cdot) \) is a natural weak equivalence.

**Example 3.2.9.** Let \( S^n \) denote the usual \( n \)-sphere in \( K \) with a basepoint \( x_0 \). For a pointed topological space \( X \), define \( \Sigma^n X = X \wedge S^n \) and \( \Omega^n X \) as \( \text{Map}_+(S^n, X) \). Similarly, for a pointed simplicial set \( X \), define \( \Sigma^n X = X \wedge \partial \Delta [n+1] \) and \( \Omega^n X = \text{Map}_+(\partial \Delta [n+1], X) \). These are called the \( n \)-fold (reduced) suspension and \( n \)-fold loop space of \( X \), respectively.

There are adjunctions in the pointed homotopy category \( [\Sigma^n X, Y]_+ = [X, \Omega^n Y]_+ \) in each case. In both cases, there is a weak equivalence \( S^n \wedge S^1 \to S^{n+1} \) and similarly in the simplicial set case (in the topological case, this is actually a homeomorphism). This implies that \( \Sigma^n X \cong \Sigma^{i+j} X \) and \( \Omega^n X \cong \Omega^{i+j} X \).
Remark 3.2.10. If $A$ and $B$ are simplicial sets, we know that $|A \times B| \approx |A| \times |B|$ and similarly $|A \wedge B| \approx |A| \wedge |B|$. Clearly $|\Delta(n+1)| \equiv S^n$. This is sufficient to prove that $|\Sigma^n A| \approx \Sigma^n |A|$.

A homework assignment asks you to show that $|\text{Map}(A, B)|$ is related by a chain of weak equivalences (in either direction) to $\text{Map}(|A|, |B|)$ provided $B$ is a Kan complex. A variation on this argument shows that $|\Omega^n Z| \approx \Omega^n |Z|$ provided $Z$ is Kan.

### 3.3 Simplicial model categories

This section is just general building up of vocabulary. It contains no real theorem.

**Definition 3.3.1.** A simplicial category is a category $\mathcal{C}$ equipped with three extra pieces of structure:

1. A simplicial enrichment: between any two objects $X, Y$ there is a simplicial set $\mathcal{S}(X, Y)$ of maps, such that $\mathcal{S}(X, Y)_0 = \mathcal{C}(X, Y)$ and satisfying the usual associative composition axioms.

2. A simplicial tensor structure: for any object $X$ and any simplicial set $K$ there is an object $X \otimes K$ in $\mathcal{C}$, functorial in both variables, and so that there is a natural isomorphism: $X \otimes (K \times K') \cong (X \otimes K) \otimes K'$.

3. A simplicial cotensor structure: for any object $X$ and any simplicial set $K$, there is a mapping object $X^K$ in $\mathcal{C}$.

These structures must further be related by natural isomorphisms

$$\text{sSet}(K, \mathcal{S}(X, Y)) \cong \mathcal{C}(X \otimes K, Y) \cong \mathcal{C}(X, Y^K).$$

**Remark 3.3.2.** In the above circumstances, $\mathcal{S}(X, Y)_n = \mathcal{C}(X \otimes \Delta[n], Y)$, so that $\mathcal{S}(X, Y)$ is determined by the rest of the structure.

Note also that $\cdot \otimes K$ and $X \otimes \cdot$ are both left adjoints, so both preserve colimits and in particular preserve initial objects.

**Definition 3.3.3.** A simplicial model category $\mathcal{C}$ is a category equipped with a simplicial structure and a model structure and so that given a cofibration $f : U \rightarrow V$ and cofibration $g : W \rightarrow X$ in $\text{sSet}$, the induced map

$$f \square g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \rightarrow V \otimes X$$

is a cofibration which is trivial if either $f$ or $g$ is.

**Remark 3.3.4.** This definition has consequences for the adjoints of $\otimes$. For instance, it follows that for a fixed cofibrant $X$, the functors $X \otimes \cdot$ and $\mathcal{S}(X, \cdot)$ form a Quillen adjunction.

**Example 3.3.5.** The model structure on $\text{sSet}$ is simplicial. Both the cotensor and the simplicial mapping object are given by $\text{Map}$.

A more interesting example is given by $K$. This is made into a simplicial model structure by defining

$$X \otimes A := X \times |A|$$

$$\mathcal{S}(X, Y)_n := K(X[n], Y)$$

$$X^A := \text{Map}(|A|, X).$$

The pointed versions of these simplical model structures also have simplicial structures. For instance, when $X$ is a pointed space, $X \otimes A$ is $X \wedge A_-$. The mapping space $\mathcal{S}(X, Y)$ is a simplicial set already (forgetting the basepoint) and $X^A$ is given by $\text{Map}_+(A_+, X)$ (which is just $\text{Map}(A, X)$, forgetting the basepoint.
Remark 3.3.6. There are model categories that are not simplicial (or at least, carry no obvious simplicial structure), but we will not encounter these in this course.
Bibliography


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