Due on 31 March 2020, v2.

(1) In this question, you may assume that the formation of the nerve $NC$ of a small category is functorial, in that a functor $h : C \to D$ induces a map of simplicial sets $Nh : NC \to ND$. Moreover, a natural transformation $\Phi : h \to h'$ induces a (left) homotopy $Nh \simeq Nh'$.

(a) Suppose $\iota : C \to D$ is an equivalence of small categories. Show that $N\iota$ is a weak equivalence of simplicial sets. (You may use any well known definition of “equivalence of categories”)

(b) Suppose $h : M \to S$ is a surjective function of sets. Form a category $C$ where the objects are the elements of $M$ and there is a unique morphism from $m$ to $m'$ if and only if $h(m) = h(m')$, and no morphisms otherwise. Prove that $NC$ is homotopy equivalent to the discrete set $S$.

(c) Let $C, \tau$ be a site with enough points. Let $X \in C$ be an object. Let $U = \{ f_i : U_i \to X \}_{i \in I}$ be a covering family. Form the nerve of the covering, i.e., the simplicial presheaf $NU$ having as $m$-th level a disjoint union of all $(m+1)$-fold products $\eta_{U_{i_0}} \times \eta_X \cdots \times \eta_X$ $\eta_{U_{i_m}}$. Prove that the evident map $NU \to X$ is a local weak equivalence.

(2) Prove that a trivial fibration $f : X \to Y$ of simplicial sets is surjective on 0-simplices (this should be easy).

(3) Suppose $X \to Y$ is the inclusion of a CW subcomplex in the category of $k$-spaces. Consider the diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
* & \text{cocone} & \\
\end{array}
$$

Prove that the natural map from the homotopy colimit to the colimit of this diagram is a weak equivalence. Hint: collapsing a contractible subcomplex of a CW complex $Z$ results in a space equivalent to $Z$. Either prove this or find a reference.

(4) Let $p \geq 2$ be a prime number. Consider the self map $f_n : S^1 \to S^1$ given by $z \mapsto z^p$. This is a map in the pointed category of CW complexes. Suppose $(X, x_0)$ is a pointed simply connected CW complex that is $\{ f_p \}$-local. Prove that $\pi_1(X, x_0)$ is uniquely $p$-divisible for all $i \geq 0$. Suppose $Y$ is an $\{ f_p \}$-local replacement for $S^2$. Describe the cohomology $\tilde{H}^*(Y, Z)$.

You may assume without proof that $\tilde{H}^*(Y, A) = [Y, K(A, i)]$ in the pointed Quillen model structure.