HOMEWORK 2

Due on 24 February 2020.

(1) Let $\mathbf{C}$ be a category with a set of objects and a set of morphisms. For a given integer $m \geq 0$, let $\mathbf{NC}_m$ be the set of sequences $c_0 \overset{f_1}{\to} c_1 \overset{f_2}{\to} c_2 \overset{f_3}{\to} \ldots \overset{f_m}{\to} c_m$ of composable morphisms in $\mathbf{C}$. The set $\mathbf{NC}_0$ consists of the objects of $\mathbf{C}$. Define face maps $d_i : \mathbf{NC}_m \to \mathbf{NC}_{m-1}$ by omitting $c_i$. In the case of $i = 0$ or $i = m$, we omit $f_1$ or $f_m$ as well. In the other cases, replace the maps $c_{i-1} \overset{f_i}{\to} c_i \overset{f_{i+1}}{\to} c_{i+1}$ by the composite $c_{i-1} \overset{f_{i+1}\circ f_i}{\to} c_{i+1}$. Define degeneracy maps $s_i : \mathbf{NC}_m \to \mathbf{NC}_{m+1}$ by adding an identity map $\text{id}_{c_i}$ in the sequence in the obvious place.

(a) Verify that $\mathbf{NC}_\bullet$ forms a simplicial set.
(b) Show that $\mathbf{NC}_\bullet$ is a quasicategory.
(c) Prove that if every morphism in $\mathbf{C}$ is an isomorphism ($\mathbf{C}$ is a groupoid), then $\mathbf{NC}_\bullet$ is a Kan complex.

(2) Suppose $X_\bullet$ is a Kan complex and $x \in X_2$ is a nondegenerate 2-simplex. Prove $X$ contains a nondegenerate 3 simplex.

(3) You may assume that for any $k$-space, the natural map $|\text{Sing } X| \to X$ is a weak equivalence.

(a) Give an example of a simplicial set $A$ and a weak equivalence of simplicial sets $B \to B'$ such that $\text{Map}(A, B) \to \text{Map}(A, B')$ is not a weak equivalence.
(b) Show that if $B$ and $B'$ are also assumed to be Kan complexes, then $\text{Map}(A, B) \to \text{Map}(A, B')$ is a weak equivalence. Deduce that $\text{Map}(A, B)$ is weakly equivalent to $\text{Map}(A, \text{Sing } |B|)$—this whole question should just be an exercise in Quillen adjunctions.
(c) Show that if $B$ is a Kan complex, then $\text{Map}(|A|, |B|)$ is weakly equivalent to $|\text{Map}(A, B)|$ (i.e., there is an isomorphism in the homotopy category between these two objects).

(4) Identify $S^1$ with the set of unit complex numbers. Let $L : \mathbf{K} \to \mathbf{K}$ denote the free loop space functor, that is, $LX = \text{Map}(S^1, X)$. Show that $LS^1 \simeq \mathbb{Z} \times S^1$. Hint A: maps $S^1 \to S^1$ admit a pointwise multiplication $fg(x) := f(x)g(x)$, which satisfies $\deg(fg) = \deg(f) + \deg(g)$ you can use singular homology to prove this. This allows you to show that $LS^1$ decomposes into homeomorphic subspaces, each corresponding to a different degree. Hint B: you can find a homotopy fibre sequence relating $LS^1$, $\Omega S^1$ and $S^1$. 

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