HOMEWORK 1

Due on 10 February 2020.

(1) Give an example, with proof, of a map of topological spaces $f : X \to Y$ that is a weak equivalence but not a homotopy equivalence. In order to prove a map is not a homotopy equivalence, it may be helpful to know about the Strøm model structure on compactly generated weak Hausdorff spaces ([BR13, Section 2]).

(2) Suppose $C$ is a model category. A pointed object of $C$ is an object $X$ and a map $x_0 : * \to X$ (a basepoint). A map of pointed objects is a map $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$. There is a category $C_+$ of pointed objects in $C$. There is a forgetful functor $U : C_+ \to C$, and a disjoint-basepoint functor $P : X \mapsto X \coprod *$. Put a model structure on $C_+$ where the weak equivalences, fibrations and cofibrations are weak equivalences, fibrations and cofibrations in $C$. You do not have to verify that this is a model structure (it is not a particularly difficult proof, [Hov99, Proposition 1.1.8]) Prove that $P, U$ form a Quillen adjunction.

(3) Let $f : R \to S$ be a map of (unital) rings, and let $\text{Ch}(R)$ and $\text{Ch}(S)$ denote the categories of nonnegatively graded chain complexes of left $R$- and $S$-modules. There exists a model structure on $\text{Ch}(R)$ where the weak equivalences are the quasi-isomorphisms, the fibrations are the levelwise surjective maps and the cofibrant objects are the complexes that are levelwise projective. This is a variant of the structure in [Hov99, Section 2.3]

There is an adjoint pair of functors $\epsilon : R\text{Mod} \leftarrow S\text{Mod} : \rho$ given by extension $\epsilon(M) = S \otimes_R M$ and restriction $\rho(M)$ of scalars. The $R$-module $\rho(M)$ agrees with $M$ as an abelian group and the $R$-structure is given by $r \cdot m = f(r)m$. Prove that $\epsilon$ and $\rho$ induce an adjoint pair of functors between $\text{Ch}(R)$ and $\text{Ch}(S)$. Prove that $\rho$ is a right Quillen functor, and describe what the total derived functor of $\epsilon$ does to objects (it’s enough to give the isomorphism class of $L\epsilon(X)$ when $X$ is a chain complex—don’t worry about functoriality of the construction).

Note: a version of this construction also makes sense for unbounded complexes, but it is harder to describe the cofibrant objects in that case, so I have not assigned this as a problem.

(4) The simplicial set $\Delta[n]$ has $\binom{n+1}{k}$ nondegenerate $k$ simplices, each corresponding to a subset of $\{0, 1, \ldots, n\}$. Consider the 0-simplices of $\Delta[n] \times \Delta[1]$. These can be identified with $\{0, 1, \ldots, n\} \times \{0, 1\}$. Write $(i, 0) = i_0$ and $(i, 1) = i_1$, so that the 0-simplices of $\Delta[n] \times \Delta[1]$ are $\{0_0, \ldots n_0, 0_1, \ldots, n_1\}$. Describe the nondegenerate $n + 1$ simplices of $\Delta[n] \times \Delta[1]$, specifically, how many are there and what are their 0-simplicies?

REFERENCES
