MATH 527/427 HOMEWORK 5

Due 3 April 2019.

Special instructions: if you are enrolled in Math 427, you will get full marks for correct answers to four out of five questions (if you submit five answers, the best four will be counted). If you are enrolled in Math 527, answer all questions.

(1) Throughout, let R be a ring.

(a) Let R be a domain, and let $r \in R \setminus \{0\}$ be a non-zerodivisor. Suppose M is a flat R-module. By applying $\cdot \otimes_R M$ to the map $\times r : R \to R$, show that the r-torsion in M is 0.

(b) Consider a sequential diagram of R-modules

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots$$

Define a map $f = \bigoplus_i f_i : \bigoplus_i M_i \to \bigoplus_i M_i$ by $f(m_1, m_2, m_3, \dots) = (0, f(m_1), f(m_2), \dots)$. Let $\operatorname{colim}_i M_i$ denote the R-module obtained as the cokernel of the map

$$1 - f : \bigoplus_i M_i \to \bigoplus_i M_i.$$

Equivalently, $\operatorname{colim}_i M_i$ can be presented as the module of all tails of sequences[1] $(m_n, f(m_n), f^2(m_n), \dots)$ with $m_n \in M_n$ modulo the relation that two sequences that eventually agree are identified.

Suppose given a commutative diagram of sequences

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "h_0"] & A_1 \arrow[r] \arrow[d, "h_1"] & A_2 \arrow[r] \arrow[d, "h_2"] & \dots \\ B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & \dots \end{tikzcd}$$

in which all maps $h_i : A_i \to B_i$ are injective. Prove that the induced map on colimits

$$h : \operatorname{colim}_i A_i \to \operatorname{colim}_i B_i$$

is injective.[2]

(c) Prove that the $\mathbb{Z}$-module $\mathbb{Q}$ is (isomorphic to) the colimit of the submodules $\{\frac{1}{n!}\mathbb{Z}\}_n$ under the evident inclusion. Hence prove that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, i.e., that $\mathbb{Q} \otimes_{\mathbb{Z}} \cdot$ preserves injections.

(2) Any abelian group A has a torsion submodule A_{tors}, consisting of those elements a such that $ma = 0$ for some $m \neq 0$. The assignment $A \mapsto A_{\text{tors}}$ is functorial, i.e., there is an induced map on homomorphisms $f \mapsto f_{\text{tors}}$ that respects compositions and identity maps.

(a) Show the inclusion $A_{\text{tors}} \to A$ is not split in general, for instance for the group

$$A = \frac{\mathbb{Z}}{(2)} \times \frac{\mathbb{Z}}{(3)} \times \frac{\mathbb{Z}}{(5)} \times \frac{\mathbb{Z}}{(7)} \times \dots.$$

This shows that abelian groups cannot be decomposed into a torsion and a nontorsion part in general.

(b) Identifying A with $\mathbb{Z} \otimes_{\mathbb{Z}} A$, show that A_{tors} lies in the kernel of $A \to \mathbb{Q} \otimes_{\mathbb{Z}} A$.

(c) Prove that $\operatorname{Tor}(\mathbb{Q}/\mathbb{Z}, A) \cong A_{\text{tors}}$. You may use the results of question (1).

[1] Indexes on f_ns have been omitted.

[2] The definitions in this question, and the conclusion, can be made equally well for any filtered indexing category, not just $0 \to 1 \to 2 \to \dots$.

(3) The following will appear in lectures. It is known as *Alexander duality* Let K be a compact, locally contractible, nonempty, proper subspace of S^n. Then there is an isomorphism

$$\tilde{\mathrm{H}}_i(S^n \setminus K; R) \cong \tilde{\mathrm{H}}^{n-i-1}(K; R).$$

(a) Let $\phi : S^1 \to \mathbb{R}^2$ be a continuous embedding (i.e., a homeomorphism of S^1 with $\phi(S^1)$). Prove that $\mathbb{R}^2 \setminus \phi(S^1)$ consists of two connected components, one bounded and the other unbounded.

(b) Let $\psi : S^1 \to \mathbb{R}^3$ be a continuous embedding. Calculate $\mathrm{H}_1(\mathbb{R}^3 \setminus \psi(S^1); \mathbb{Z})$.

(4) If X is a topological space and $f : X \to X$, we can produce the *mapping torus* of f, denoted MT_f as the quotient of the space $X \times [0,1]$ by the relation $(x,0) \sim (f(x),1)$ for all $x \in X$. When X is an n-manifold and f is a homeomorphism, MT_f is an $n+1$-manifold.

(a) Using the Mayer–Vietoris sequence or otherwise, establish a long exact sequence in cohomology

$$\ldots \longrightarrow \mathrm{H}^i(MT_f; R) \longrightarrow \mathrm{H}^i(X; R) \xrightarrow{\mathrm{id}-f} \mathrm{H}^i(X; R) \longrightarrow \mathrm{H}^{i+1}(MT_f; R) \longrightarrow \ldots$$

(b) The cohomology of the torus $T = S^1 \times S^1$ can be described as the ring

$$\mathrm{H}^*(T; R) = \Lambda_R(e_1, e_2), \quad |e_1| = |e_2| = 1.$$

That is, it is the graded R-algebra generated by two degree-1 elements, e_1 and e_2, subject to the relations $e_1^2 = e_2^2 = 0$ and $e_1 e_2 = -e_2 e_1$.

Fix $\{e_1, e_2\}$ as a basis for $\mathrm{H}^1(T; \mathbb{Z})$. Suppose $f : T \to T$ is a homeomorphism, then $f^* : \mathrm{H}^1(T; \mathbb{Z}) \to \mathrm{H}^1(T; \mathbb{Z})$ is an element $A \in \mathrm{GL}_2(\mathbb{Z})$. Calculate $f^* : \mathrm{H}^2(T; \mathbb{Z}) \to \mathrm{H}^2(T; \mathbb{Z})$.

(c) Let $f : T \to T$ be a homeomorphism and let A be as in the previous part. Describe $\mathrm{H}^*(MT_f; \mathbb{Z})$ in terms of[3] A. Under what circumstances is MT_f an orientable manifold?

(5) Let $n \geq 3$. Recall that S^n carries an antipodal action $(x_0, \ldots, x_n) \mapsto (-x_0, \ldots, -x_n)$. The quotient space is $\mathbb{R}P^n$. Let $\gamma : I \to S^n$ be a path for which $\gamma(0) = -\gamma(1)$. Then covering space theory says that the composite $\gamma : I \to S^n \to \mathbb{R}P^n$ represents a generator of $\pi_1(\mathbb{R}P^n, *) \cong \mathbb{Z}/(2)$.

(a) Prove using $\mathrm{H}^*(\cdot; \mathbb{F}_2)$ that there is no map $\mathbb{R}P^n \to \mathbb{R}P^{n-1}$ inducing an isomorphism on fundamental groups.

(b) Suppose $f : S^n \to \mathbb{R}^n$ is a continuous function such that $f(-\mathbf{x}) = -f(\mathbf{x})$. Prove there is some $\mathbf{v} \in S^n$ such that $f(\mathbf{v}) = \mathbf{0}$.

(c) By an *oriented hyperplane* in $\mathbb{R}^n$, we mean a plane P, not necessarily passing through the origin, and a selection of one of the two connected components of $\mathbb{R}^n \setminus P$ to be a *positive half-space*, U^+. The other component is the *negative half space*, U^-. An oriented plane is equivalent to the data of a hyperplane along with a distinguished choice of unit normal vector.

There is a concept of volume in $\mathbb{R}^n$. We will simply call this *measure*. You may assume measure is additive, $\mu(A \cup B) = \mu(A) + \mu(B)$ if A, B are disjoint, and that the measure $\mu(P)$ of a hyperplane $P \subset \mathbb{R}^n$ is 0. You may assume all objects appearing in this exercise have a well defined nonnegative measure, possibly 0.

Let X_0 be a closed, bounded, convex subset of $\mathbb{R}^n$ of positive measure. We say P *bisects* X_0 if $\mu(X_0 \cap U^+) = \mu(X_0 \cap U^-) = \mu(X_0)/2$. Let $\mathbf{v} \in S^n$ be a vector. Explain why there is an oriented plane $P_{\mathbf{v}}$ having unit normal $\mathbf{v}$ and bisecting X_0 (note: this is not an exercise in measure theory, so you don't have to justify this rigorously).

(d) Let $G_{n-1}(n)^o$ denote the space of all oriented planes in $\mathbb{R}^n$—it is homeomorphic to $S^n \times \mathbb{R}$. Let X_0 be as in the previous part. You may assume the assignment $\phi : S^n \to G_{n-1}(n)^o$ sending $\mathbf{v}$ to $P_{\mathbf{v}}$ defines a continuous function.

Let $X_1, \ldots, X_{n-1}$ be subspaces of $\mathbb{R}^n$ of well defined measure. Let X_0 be as in the previous part. Define a function $\phi : G_{n-1}(n)^o \to \mathbb{R}^{n-1}$ sending an oriented hyperplane P with positive half-space U^+ and negative half space U^- to

$$\big(\mu(U^+ \cap X_1) - \mu(U^- \cap X_1), \mu(U^+ \cap X_2) - \mu(U^- \cap X_2), \ldots, \mu(U^+ \cap X_{n-1}) - \mu(U^- \cap X_{n-1})\big)$$

You may assume ϕ is continuous. Prove that there exists some hyperplane P in $\mathbb{R}^n$ such that P bisects each of $X_0, X_1, \ldots, X_{n-1}$ simultaneously.

[3] Such expressions as $\mathbb{Z}^2/(\mathrm{Im}(I_2 - A))$ are acceptable

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