(1) (a) Suppose given a commutative diagram of $R$-modules

\[ \ldots \rightarrow C_{n+1} \xrightarrow{h_{n+1}} A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \xrightarrow{h_n} A_{n-1} \xrightarrow{\gamma_n} C_{n+1} \xrightarrow{h'_{n+1}} A'_n \xrightarrow{f'_n} B'_n \xrightarrow{g'_n} C'_n \xrightarrow{h'_n} A'_{n-1} \xrightarrow{\gamma'_n} C'_{n+1} \xrightarrow{h'_{n+1}} \ldots \]

in which the maps $\alpha$ are isomorphisms and in which the rows are long exact sequences. Define maps $\partial : C'_n \rightarrow A_{n-1}$ by $\partial = f'_{n-1} \circ \alpha_{n-1}^{-1} \circ h'_n$. Prove that the sequence

\[ \ldots \rightarrow B_n \xrightarrow{\beta_n \circ g_n} B'_n \xrightarrow{\alpha_n^{-1}} C_n \xrightarrow{\partial} A_{n-1} \rightarrow \ldots \]

is exact.

(b) Suppose $A$ and $B$ are two subcomplexes of a CW complex $X$ such that $X = A \cup B$. You may use the fact $X/A = B/(A \cap B)$. Use the previous part of this question to establish the Mayer–Vietoris long exact sequence:

\[ \cdots \rightarrow H_n(A \cap B; R) \rightarrow H_n(A; R) \oplus H_n(B; R) \rightarrow H_n(X; R) \rightarrow H_{n-1}(A \cap B; R) \rightarrow \cdots \]

(explaining what the maps are).

(2) Read the example on page 5, chapter 0 of Hatcher explaining how to produce the genus-$g$ surface as a CW complex. That is, a genus-$g$ surface $M_g$ is obtained by identifying edges in a convex 4-gon in alternate pairs with reversed orientation on the edges. Calculate the homology $H_*(M_g; R)$.

(3) Consider the map $f : \mathbb{R}P^2 \rightarrow S^2$ given by collapsing the 1-skeleton $\mathbb{R}P^1 \subset \mathbb{R}P^2$ to a point. Calculate $f_* : H_*(\mathbb{R}P^2; \mathbb{Z}) \rightarrow H_*(S^2; \mathbb{Z})$ and $f_* : H_*(\mathbb{R}P^2; \mathbb{Z}/2) \rightarrow H_*(S^2; \mathbb{Z}/2)$

(4) (a) Show that the degree of the antipodal $a : S^n \rightarrow S^n$ map given by $a(v) = -v$ is $(-1)^{n+1}$.

(b) Show that if $f : S^n \rightarrow S^n$ is a map of spheres without a fixed point, that $f \equiv a$.

(c) Suppose $g : D^n \rightarrow D^n$ is a self-map of the closed $n$-disk. Construct $S^n$ by identifying two copies $D^n_+$ and $D^n_-$ of $D^n$ using the identity map on the boundary $S^{n-1}$. Define a map $G : S^n \rightarrow S^n$ by $G(d) = g(d) \in D^n_+$ for $d \in D^n_-$ and also for $d \in D^n_+$. That is, both hemispheres are mapped to the hemisphere $D^n_+$. This map is continuous since the piecewise definition agrees on the boundary $S^{n-1}$. Prove that $G$ has a fixed point, and therefore that $g$ has a fixed point. This is Brouwer’s fixed point theorem.

(5) Produce a CW complex as follows. Start with a solid cube, with the the polyhedral CW structure: there are eight 0-cells at the vertices, twelve 1-cells, being the edges, six 2-cells being the faces and a single interior 3-cell. Now form the quotient CW complex $Q$ by identifying antipodal (closed) faces, but with a twist. For each pair of antipodal closed faces $T$ and $B$, rotate the cube so $T$ is on top and $B$ on the bottom. Then identify $T$ with $B$ by twisting $T$ by $\pi/2$ in the positive direction, and then projecting downwards. (Observe that the identification specified by this “anticlockwise twist” is the same whether $T$ or $B$ is placed on top.)
The resulting CW complex $Q$ has one 3-cell, three 2-cells and some number of 1- and 0-cells. Determine the cell structure of $Q$ and calculate $H_*(Q; \mathbb{Z})$. 