MATH 426, TOPOLOGY
THE $p$-NORMS

In this document we assume an extended real line, where $\infty$ is an element greater than all real numbers; the interval notation $[1, \infty]$ will be used to mean $[1, \infty) \cup \{\infty\}$.

1. THE $p$ NORMS ON $\mathbb{R}^n$

Fix an integer $n \geq 1$. When $p \geq 1$ is a real number, we define

$$\|x\|_p = \left( \sum_{i=0}^{n} |x_i|^p \right)^{1/p}. $$

Define

$$\|x\|_{\infty} = \sup_i |x_i|. $$

For a given real number $p > 1$, the Hölder conjugate of $p$ is the number $q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1;$$

this is equivalent to

$$q = \frac{p}{p - 1}. $$

Another equivalent formulation is

$$qp - q - p = 0$$

Observe that 2 is self-conjugate, but no other number is. We also declare the pair $\{1, \infty\}$ to be Hölder conjugates.

Each of these norms has the following property: given any $x \in \mathbb{R}^n$ and any $r \in \mathbb{R}$, we have

$$\|rx\|_p = |r|\|x\|_p.$$ 

This will be important later.

It is also immediate, $p \in [1, \infty)$, that $\|x\|_p = 0$ if and only if $x = 0$.

**Proposition 1.1** (Young’s Inequality). Let $p, q$ be a Hölder conjugate pair in $(1, \infty)$ and suppose $a, b$ are nonnegative real numbers, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with equality if and only if $a^p = b^q$.

**Proof.** Define

$$f(x) = \frac{x^p}{p} + \frac{b^q}{q} - bx.$$ 

Then $f'(x) = x^{p-1} - b$ and $f''(x) = (p-1)x^{p-2}$. We observe that since $p > 1$, the function $f(x)$ is convex on $(0, \infty)$ and any critical point will be a global minimum on that interval. There is a unique critical point, when $x = b^{1/(p-1)}$, which is to say when $x^p = b^q$ by (1).
Evaluating at this value, we obtain \( f(b^{1/(p-1)}) = 0 \). Therefore \( f(x) \geq 0 \), with equality exactly when \( x^p = b^p \), which was what we wanted to prove.

**Proposition 1.2 (Hölder’s Inequality).** For a given \( p \in [1,\infty] \), having Hölder conjugate \( q \), and any two vectors \( x, y \in \mathbb{R}^n \), one has

\[
\sum_{i=1}^n |x_i y_i| \leq \|x\|_p \|y\|_q.
\]

**Proof.** When \( p = 1 \) and \( q = \infty \), or vice versa, this amounts to the triangle inequality for the absolute value on \( \mathbb{R}^1 \).

We therefore assume \( 1 < p < \infty \). By referring to (3), we see that it suffices to prove the proposition after replacing \( x \) and \( y \) by \( rx \) and \( sy \) where \( 0 < r \) and \( 0 < s \), so we may assume that \( \|x\|_p = \|y\|_q = 1 \).

By repeated use of Young’s inequality, we obtain the inequality

\[
\sum_{i=1}^n |x_i y_i| \leq \frac{\sum_{i=1}^n |x_i|^p}{p} + \frac{\sum_{i=1}^n |y_i|^q}{q} = \frac{\|x\|_p^p}{p} + \frac{\|y\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 = \|x\|_p \|y\|_q.
\]

**Proposition 1.3 (Minkowski Inequality).** Let \( p \in [1,\infty] \) and \( x, y \in \mathbb{R}^n \), then

\[
\|x + y\|_p \leq \|x\|_p + \|y\|_p.
\]

**Proof.** The cases of \( p = 1 \) and \( p = \infty \) reduce immediately to the usual triangle inequality for \( \mathbb{R} \).

Assume that \( 1 < p < \infty \).

Consider a vector \( w \) having \( i \)-th coordinate \( w_i = |x_i + y_i|^{p-1} \). We calculate

\[
\|w\|_q = \left( \sum_{i=1}^n |x_i + y_i|^{q(p-1)} \right)^{1/q} = \left( \sum_{i=1}^n |x_i + y_i|^q \right)^{1/q} = \|x + y\|^{p/q}_p.
\]

Now we split up \( \|x + y\|_p \) as follows:

\[
\|x + y\|_p = \sum_{i=1}^n |x_i + y_i|^p \leq \sum_{i=1}^n (|x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1}) \leq \|x\|_p \|w\|_q + \|y\|_p \|w\|_q
\]

where the last inequality is the Hölder inequality. We have a formula for \( \|w\|_q \), which we use to deduce

\[
\|x + y\|_p \leq \|x\|_p \|w\|_q + \|y\|_p \|w\|_q = (\|x\|_p + \|y\|_p)\|x + y\|^{p/q}_p.
\]

Dividing through gives

\[
\|x + y\|_p \leq \|x\|_p + \|y\|_p,
\]

which is what we wanted.

**Exercise 1.4.** Show that for a given vector \( x \in \mathbb{R}^n \), the function \( p \to \|x\|_p \) is decreasing on \( p \in [1,\infty] \).

Show further that \( \lim_{p\to\infty} \|x\|_p = \|x\|_\infty \).

**Proposition 1.5.** Given \( x \in \mathbb{R}^n \) and any \( p \in [1,\infty] \)

\[
\|x\|_1 \geq \|x\|_p \geq \|x\|_\infty \geq \frac{1}{n} \|x\|_1.
\]
This follows from Exercise 1.4 and the observation that \( \|x\|_1 \leq n\|x\|_\infty \).

**Proposition 1.6.** Given any \( \{p, q\} \subset [1, \infty] \), there exist constants \( c, C > 0 \), such that for any \( x \in \mathbb{R}^n \), we have

\[
C\|x\|_p \geq \|x\|_q \geq c\|x\|_p.
\]

This follows immediately from 1.5. It may be worthwhile to find the best possible constants \( c, C \), but we will not do that here.

## 2. Norms and metrics

**Definition 2.1.** A real normed linear space will consist of a \( \mathbb{R} \) vector space \( V \) and a norm \( \| \cdot \| : V \to \mathbb{R} \) with the following properties. For all \( v, w \in V \) and \( r \in \mathbb{R} \):

1. \( \|v\| \geq 0 \), with equality if and only if \( v = 0 \).
2. \( \|rv\| = |r|\|v\| \).
3. \( \|v + w\| \leq \|v\| + \|w\| \).

An obvious complex analogue of the above also may be defined.

**Proposition 2.2.** For any \( n \in \mathbb{N} \) and any \( p \in [0, \infty] \), the pair \( (\mathbb{R}^n, \| \cdot \|_p) \) defined in the previous section is a normed linear space.

**Proof.** Easy. \( \square \)

**Proposition 2.3.** If \( (V, \| \cdot \|) \) is a normed linear space, then the function \( d(v, w) = \|v - w\| \) defines a metric on \( V \).

**Proof.** This is not at all difficult.

1. Property 1 of Definition 2.1 implies immediately that \( d(x, y) \geq 0 \) with equality if and only if \( x = y \).
2. Property 2 of Definition 2.1 with \( r = -1 \) shows that

\[
d(x, y) = \|x - y\| = \|y - x\| = d(y, x).
\]

3. Property 3 of Definition 2.1 applies to give

\[
d(x, y) = \|x - y\| = \|(x - z) - (y - z)\| \leq d(x, z) + d(y, z).
\]

\( \square \)

**Notation 2.4.** The notation \( d_p \) is used for the metric associated to the normed linear space \( (\mathbb{R}^n, \| \cdot \|_p) \).

## 3. The \( p \)-norms and product metrics

**Notation 3.1.** The notation \( (x_n) \) will be used to denote a sequence (finite or infinite) of real numbers indexed by a natural number \( n \). So \( (x_n) \) means the same thing as \( (x_1, x_2, x_3, \ldots) \), finite or infinite depending on context. Occasionally, we will have a need to write something complicated like the sequence \( (m, m/2, m/3, \ldots) \) where there is a parameter. In this case we may write the sequence as \( (m/n)_n \), where the external ‘\( n \)’ indicates that \( n \) is the variable indexing the terms of the sequence.

**Definition 3.2.** Let \( ((X_1, d_1), \ldots, (X_n, d_n)) \) be a finite set of metric spaces. Let \( X = \prod_{i=1}^n X_i \), let \( p \in [1, \infty] \). Define a function \( d^p : X \times X \to [0, \infty) \) by \( d^p((x_i), (y_i)) = \|(d_i(x_i, y_i))\|_p \).

**Proposition 3.3.** The functions \( d^p \) defined above are all metrics.
Proof. Symmetry is immediate. If \( d^p((x_i), (y_j)) = \|d_i(x_j, y_i)\|_p = 0 \), then \( d_i(x_j, y_i) = 0 \) for all \( i \), and since \( d_i \) is a metric, this implies \( (x_i) = (y_j) \). The triangle inequality is given by combining the triangle inequalities for each \( d_i \) metric with that for \( \| \cdot \|_p \), and noting that \( \| \cdot \|_p \) is increasing in each variable:

\[
d^p((x_i), (y_j)) = \|d_i(x_j, y_i)\|_p \leq (\|d_i(x_i, z_i)\|_p + \|d_i(z_i, y_j)\|_p)
\leq \|d_i(x_i, z_i)\|_p + \|d_i(z_i, y_j)\|_p = d^p((x_i), (z_i)) + d^p((z_i), (y_j))
\]

\( \square \)

Proposition 3.4. The metrics \( d^p \) all generate the same topologies.

Proof. It suffices to show that for any \( p, p' \in [1, \infty] \), any ball \( B_p((x_i), \epsilon) \) for the \( d^p \) metric with \( \epsilon > 0 \) contains a ball \( B_{p'}((x_i), \eta) \) for the \( d^{p'} \) metric with \( \eta > 0 \) and having the same centre.

We know from Proposition 1.6 that there is some constant \( c > 0 \) such that \( c\|(d(x_i, y_j))\|_p \leq \|d(x_i, y_j)\|_{p'} \). Then \( B_{p'}((x_i), ce) \subseteq B_p((x_i), \epsilon) \).

Exercise 3.5 (Done in lecture). The metric \( d^\infty \) generates the product the topology; therefore all the metrics \( d^p \) generate the product topology.

Remark 3.6. It is easily seen that \( d_p \) on \( \mathbb{R}^n \) from Notation 2.4 is the product metric \( d^p \) for \( n \) copies of \( (\mathbb{R}, | \cdot |) \). By reference to Proposition 3.4, the metric spaces \( (\mathbb{R}^n, d_p) \) and \( (\mathbb{R}^n, d_{p'}) \) all generate equivalent topologies for all \( p, p' \in [1, \infty] \), and this topology is the product topology on \( \mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} \).

4. THE \( p \)-NORMS ON SEQUENCE SPACES

Definition 4.1. If \( p \in [1, \infty) \), we define a set \( \ell^p \subseteq \prod_{i=1}^\infty \mathbb{R} \) to consist of those sequences \( (x_n) \) such that

\[
\sum_{i=1}^\infty |x_i|^p
\]

converges to a real number. For such a sequence, we define

\[
\|(x_n)\|_p = \left( \sum_{i=1}^\infty |x_i|^p \right)^{1/p}.
\]

Proposition 4.2. The pair \( (\ell^p, \| \cdot \|_p) \) is a normed linear space.

Proof. Conceptually, in the first place we must show that \( \ell^p \) is a vector subspace of \( \prod_{i=1}^\infty \mathbb{R} \). We must show it is closed under addition of vectors and under scalar multiplication. In the second, we must show that \( \| \cdot \|_p \) has the properties of a norm. In practice, it is simpler to prove all these facts concerning addition together then all the facts concerning scalar multiplication.

Suppose \( (x_n) \) and \( (y_n) \) are sequences in \( \ell^p \), then for all \( N \in \mathbb{N} \)

\[
\left( \sum_{i=1}^N |x_i + y_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^N |x_i|^p \right)^{1/p} + \left( \sum_{i=1}^N |y_i|^p \right)^{1/p}
\]

by the Minkowski inequality.

Rearranging this, we deduce

\[
\sum_{i=1}^N |x_i + y_i|^p \leq (\|(x_n)\|_p + \|(y_n)\|_p)^p.
\]

The right hand side is the \( N \)-th partial sum of the series

(4)

\[
\sum_{i=1}^\infty |x_i + y_i|^p
\]
which consists of positive terms. The left hand side is independent of \( N \), and therefore we deduce that (4) converges, and the limit is less than or equal to \( \| (x_n) \|_p + \| (y_n) \|_p^p \). Rearranging, we deduce that

- \( (x_n) + (y_n) \in \ell^p \)
- \( \| (x_n) + (y_n) \|_p \leq \| (x_n) \|_p + \| (y_n) \|_p \).

As for scalar multiplication, it is straightforward to show that

\[ \| r(x_n) \|_p = \left( \sum_{i=1}^{\infty} |r x_i|^p \right)^{1/p} = |r| \| (x_n) \|_p \]

which shows that

- \( r(x_n) \in \ell^p \),
- \( \| r(x_n) \|_p = |r| \| (x_n) \|_p \).

Finally, we observe that \( \| (x_n) \| = 0 \) if and only if every term of \( (x_n) \) is 0.

**Definition 4.3.** We define \( \ell^\infty \) to consist of those sequences \( (x_n) \) such that \( \sup_{i \in \mathbb{N}} |x_i| < \infty \), i.e. the bounded sequences. We define \( \| (x_n) \|_\infty = \sup_{i \in \mathbb{N}} |x_i| \).

**Exercise 4.4.** With the definitions above \((\ell^\infty, \| \cdot \|_\infty)\) forms a normed linear space.

**Definition 4.5.** Let \( c \) denote the set of convergent sequences of real numbers, \( c_0 \) the set of sequences of real numbers with limit 0, and \( \mathbb{R}^\infty \) or \( c_0 \) the set of sequences of real numbers having at most finitely many nonzero terms.

Unless otherwise specified, we give \( \ell^p \) the topology induced by the (metric induced by the) norm \( \| \cdot \|_p \). We give \( c \) and \( c_0 \) the topologies inherited from \( \ell^\infty \). Which norm, metric or topology one should place on \( \mathbb{R}^\infty \) is less clear, see Exercise 4.13.

**Proposition 4.6.** Suppose \( p, q \in [1, \infty) \) satisfy \( p < q \). Then there are strict inclusions

\[ \mathbb{R}^\infty \subset \ell^p \subset \ell^q \subset c_0 \subset c \subset \ell^\infty \]

and if \( (x_n) \in \ell^p \), then \( \| (x_n) \|_p \geq \| (x_n) \|_q \geq \| (x_n) \|_\infty \).

**Proof.** We prove this in several steps:

1. \( \mathbb{R}^\infty \subset \ell^1 \). The inclusion is immediate, and considering the sequence \((x_n) = (1/2, 1/4, 1/8, \ldots)\) for which \( \| (x_n) \|_1 = 1 \) but that is not in \( \mathbb{R}^\infty \) shows that it is strict.
2. Suppose \( p < q \in [1, \infty) \). Suppose \((x_n) \in \ell^p \). For any initial sequence, we have

\[ \sum_{i=1}^{N} |x_i|^q \leq \left( \sum_{i=1}^{N} |x_i|^p \right)^{q/p} \]

since \( x \to \| x \|_p \) for \( x \in \mathbb{R}^N \) is decreasing as a function of \( p \). But this implies that, in the limit,

\[ \sum_{i=1}^{\infty} |x_i|^q \leq \| (x_n) \|^q_p, \]

from which the inclusion \( \ell^p \subset \ell^q \) and the inequality \( \| (x_n) \|_p \geq \| (x_n) \|_q \) both follow.

We observe that if \( x_n = 1/\sqrt{n} \), then \( \sum_{i=1}^{\infty} |x_i|^p = \sum_{i=1}^{\infty} 1/\sqrt{n} \) diverges but \( \sum_{i=1}^{\infty} |x_i|^q = \sum_{i=1}^{\infty} 1/\sqrt{n} \) converges, both by the integral test for convergence. So the inclusion is strict.

3. If \( (x_n) \in \ell^q \), then the series \( \sum_{i=1}^{\infty} |x_i|^q \) converges, so \( \lim_{i \to \infty} x_i = 0 \), so \( (x_n) \in c_0 \). The sequence \( x_n = 1/\sqrt{n} \) shows that the inclusion is strict.

4. Any sequence converging to 0 converges, so \( c_0 \subset c \). The inclusion is clearly strict, since the constant function 1 converges, but not to 0.
Corollary 4.7. If $p < q$ are elements of $[1, \infty]$, then the inclusions $\ell^p \subset \ell^q$ are continuous.

Proof. These are metric spaces, and it suffices to give an $\epsilon - \delta$ proof of continuity. Let $\epsilon > 0$ and choose
$$
\delta = \epsilon. \text{ For any } (x_n), (y_n) \in \ell^p, \text{ if }
$$
$$
\| (x_n) - (y_n) \|_p < \delta
$$
then
$$
\| (x_n) - (y_n) \|_q < \delta = \epsilon.
$$
So the inclusion is indeed continuous.

Another useful fact is the following. Each space appearing in Proposition 4.6 is a subspace of the space of all sequences: $\mathbb{R}^\mathbb{N}$. This space may be equipped with the product topology.

Proposition 4.8. Let $p \in [1, \infty]$. Then the inclusion $\ell^p \rightarrow \mathbb{R}^\mathbb{N}$ is continuous.

Proof. The product topology on $\mathbb{R}^\mathbb{N}$ is the weakest topology making each projection continuous. In particular, by Proposition 3.16 from the class lecture notes, if the functions $\pi_i : \ell^p \rightarrow \mathbb{R}$ that take a sequence $(x_n)$ to the $i$-th term $x_i$ are continuous, then the induced map $\ell^p \rightarrow \mathbb{R}^\mathbb{N}$ is continuous, and it is easy to verify that this map is indeed the inclusion.

So it suffices to show that $\pi_i$ is continuous. Since the source and target are both metric spaces, an $\epsilon - \delta$ argument applies. Suppose $\| (x_n) - (y_n) \|_p < \epsilon$, then in particular $|x_i - y_i| < \epsilon$, which implies $|\pi_i((x_n)) - \pi_i((y_n))| < \epsilon$. So $\pi_i$ is indeed continuous, and the result follows.

Proposition 4.9. Let $p \in [0, \infty)$. The subset $\mathbb{R}^\infty$ is dense in $\ell^p$.

Proof. Let $(x_n)$ be a sequence in $\ell_p$, and for $m \in \mathbb{N}$ let $(x_{m,n})_n$ denote the sequence for which $x_{m,n} = x_n$ if $m \leq n$ and $x_{m,n} = 0$ otherwise.

We wish to show that $\ell^p = \mathbb{R}^\infty$. We show that every element of $\ell^p$ is a limit of a sequence of elements in $\mathbb{R}^\infty$—this is a sequence of sequences.

Let $\epsilon > 0$. Consider the sequence $(x_{m,n})_m$ of elements of $\mathbb{R}^\infty$. Observe that
$$
\| (x_{m,n})_n - (x_n) \|_p^p = \sum_{i=1}^m |x_i - x_i|^p + \sum_{i=m+1}^\infty |x_i|^p = \sum_{i=m+1}^\infty |x_i|^p.
$$
Since the series $\sum_{i=1}^\infty |x_i|^p$ converges to $\| (x_n) \|_p^p$, we can find some $N$ such that
$$
\left| \sum_{i=1}^m |x_i|^p - \| (x_n) \|_p^p \right| < \epsilon^p
$$
whenever $m > N$, which is equivalent to
\[ \sum_{i=1}^{m} |x_i|^p - \sum_{i=1}^{\infty} |x_i|^p < \epsilon^p, \]
but this is equivalent to saying
\[ \sum_{i=m+1}^{\infty} |x_i|^p < \epsilon^p \]
whenever $m > N$, which is to say that $\| (x_{m,n})_n - (x_n)_n \|_p < \epsilon^p$, and taking $p$-th roots, we see that
\[ \| (x_{m,n})_n - (x_n)_n \|_p < \epsilon \]
whenever $m > N$. Therefore the sequence $((x_{m,n}))_m \to (x_n)_m$ as $m \to \infty$. \qed

Exercise 4.10. Let $p \in [1, \infty)$. Let $Q^\infty \subset \mathbb{R}^\infty$ denote the set of sequences having only rational-number terms and which are eventually 0.

1. Prove $Q^\infty$ is dense in $\ell^p$.
2. Give $c_0$ the subspace topology inherited from $\ell^\infty$. Prove $Q^\infty$ is dense in $c_0$.

Corollary 4.11. Since $Q^\infty$ is in bijection with the countable union $\bigcup_{i=1}^{\infty} Q^i$ of countable spaces, it follows that each of the spaces $\ell^p$ for $p \in [1, \infty)$ or $c_0$ or $c$ is separable, and since they are metric, they are second countable.

Exercise 4.12. Prove that $\ell^\infty$ is not separable (and therefore, not second countable).

The situation for infinite-dimensional spaces is therefore much more complicated than for finite-dimensional spaces. In the finite-dimensional setting, there was only one linear space for each dimension, $\mathbb{R}^n$, and each of the norms $\| \cdot \|_p$ induced the same topology. In the infinite-dimensional case, the topologies and spaces on which they are defined are all different.

It can be conceptually helpful to view the elements of $\mathbb{R}^\infty$, that is, finite sequences of some undetermined length, as the objects one is most likely to encounter in practical situations; the real world is generally finitist. Then the different spaces $\ell^p$, $c_0$ and $c$ are different choices of which sequences of elements in $\mathbb{R}^\infty$ one views as convergent.

Exercise 4.13. Consider the various normed linear spaces $(\mathbb{R}^\infty, \| \cdot \|_p)$ for $p \in [1, \infty)$. Prove that these are pairwise inequivalent as metric spaces by considering which sequences of elements in $\mathbb{R}^\infty$ are convergent for the various $d_p$ metrics.

4.1. Completeness.

Exercise 4.14. Prove that the spaces $(\ell^p, \| \cdot \|_p)$ are complete for all $p \in [1, \infty]$. What can be said about $\mathbb{R}^\infty$, $c_0$ and $c$?

5. The $p$-norms for functions

This is not a course in measure theory, so we content ourselves with the following inadequate treatment.

Definition 5.1. Let $p \in [1, \infty)$. Suppose $f : [a, b] \to \mathbb{R}$ is a function defined on an interval $[a, b]$ for which
\[ \int_{a}^{b} |f|^p \, dx \]
is defined (and finite). Then define
\[ \| f \|_p = \left( \int_{a}^{b} |f|^p \right)^{1/p}. \]
Remark 5.2. The integral should really be taken in the sense of Lebesgue, but we will restrict our attention to piecewise continuous functions on closed bounded intervals, which will allow us to use only Riemann integrals, including improper Riemann integrals if necessary.

Exercise 5.3. Let $P[a, b]$ denote the set of piecewise-continuous functions on the closed bounded interval $[a, b]$. For $f \in P[a, b]$ and $p \in [1, \infty)$, show that $(P[a, b], \| \cdot \|_p)$ makes $P[a, b]$ a normed linear space.

Definition 5.4. We say that $C$ is an essential supremum for a piecewise continuous function $f : [a, b] \to \mathbb{R}$ if the set $S$ of values $x$ for which $f(x) > C$ does not have any interior points—i.e., $S$ contains no open intervals.

Exercise 5.5. For $f \in P[a, b]$, define $\| f \|_{\infty}$ to be

$$\| f \|_{\infty} = \inf\{ C : C \text{ is an essential supremum for } |f| \}.$$  

Prove that $(P[a, b], \| \cdot \|_{\infty})$ is a normed linear space.

Exercise 5.6. Show that the spaces $(P[a, b], \| \cdot \|_p)$ as $p$ varies over $[1, \infty]$ are all different.