THE $p$-NORMS: VERSION 1

In this document we assume an extended real line, where $\infty$ is an element greater than all real numbers; the interval notation $[1, \infty]$ will be used.

1. The $p$ Norms on $\mathbb{R}^n$

Fix an integer $n \geq 1$. When $p \geq 1$ is a real number, we define

$$\|x\|_p = \left( \sum_{i=0}^{n} |x_i|^p \right)^{1/p}.$$  

Define

$$\|x\|_\infty = \sup_i |x_i|.$$  

For a given real number $p > 1$, the Hölder conjugate of $p$ is the number $q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1;$$

this is equivalent to

(1) \[ q = \frac{p}{p-1}. \]

Another equivalent formulation is

(2) \[ qp - q - p = 0 \]

Observe that 2 is self-conjugate, but no other number is. We also declare the pair $\{1, \infty\}$ to be Hölder conjugates.

Each of these norms has the following property: given any $x \in \mathbb{R}^n$ and any $r \in \mathbb{R}$, we have

(3) \[ \|rx\|_p = |r| \|x\|_p. \]

This will be important later.

It is also immediate, for all applicable $p$, that $\|x\|_p = 0$ if and only if $x = 0$.

**Proposition 1.1** (Young’s Inequality). Let $p, q$ be a Hölder conjugate pair in $(1, \infty)$ and suppose $a, b$ are nonnegative real numbers, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with equality if and only if $a^p = b^q$.

**Proof.** Define

$$f(x) = \frac{x^p}{p} + \frac{b^q}{q} - bx.$$  

Then $f'(x) = x^{p-1} - b$ and $f''(x) = (p-1)x^{p-2}$. We observe that since $p > 1$, the function $f(x)$ is convex on $(0, \infty)$ and any critical point will be a global minimum on that interval. There is a unique critical point, when $x = b^{1/(p-1)}$, which is to say when $x^p = b^q$ by (1).

Evaluating at this value, we obtain $f(b^{1/(p-1)}) = 0$. Therefore $f(x) \geq 0$, with equality exactly when $x^p = b^q$, which was what we wanted to prove.  

\[ \Box \]
Proposition 1.2 (Hölder’s Inequality). For a given \( p \in [1, \infty) \), having Hölder conjugate \( q \), and any two vectors \( x, y \in \mathbb{R}^n \), one has

\[
\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q.
\]

Proof. When \( p = 1 \) and \( q = \infty \), or vice versa, this amounts to the triangle inequality for the absolute value on \( \mathbb{R}^1 \).

We therefore assume \( 1 < p < \infty \). By referring to (3), we see that it suffices to prove the proposition after replacing \( x \) and \( y \) by \( rx \) and \( sy \) where \( 0 < r \) and \( 0 < s \), so we may assume that \( \|x\|_p = \|y\|_q = 1 \).

By repeated use of Young’s inequality, we obtain the inequality

\[
\sum_{i=1}^{n} |x_i y_i| \leq \sum_{i=1}^{n} \left( \frac{|x_i|^p}{p} + \frac{|y_i|^q}{q} \right),
\]

which is

\[
\sum_{i=1}^{n} |x_i y_i| \leq \frac{1}{p} \sum_{i=1}^{n} |x_i|^p + \frac{1}{q} \sum_{i=1}^{n} |y_i|^q = \frac{1}{p} + \frac{1}{q} = 1 = \|x\|_p \|y\|_q.
\]

\( \square \)

Proposition 1.3 (Minkowski Inequality). Let \( p \in [1, \infty) \) and \( x, y \in \mathbb{R}^n \), then

\[
\|x + y\|_p \leq \|x\|_p + \|y\|_p.
\]

Proof. The cases of \( p = 1 \) and \( p = \infty \) reduce immediately to the usual triangle inequality for \( \mathbb{R} \).

Assume that \( 1 < p < \infty \).

Consider a vector \( w \) having \( i \)-th coordinate \( w_i = |x_i + y_i|^{p-1} \). We calculate

\[
\|w\|_p = \left( \sum_{i=1}^{n} |x_i + y_i|^{q(p-1)} \right)^{1/q} = \left( \sum_{i=1}^{n} (|x_i + y_i|^{p})^{q/p} \right)^{1/q} = \|x + y\|_p^{p/q}.
\]

Now we split \( \|x + y\|_p^p \) as follows:

\[
\|x + y\|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p \leq \sum_{i=1}^{n} (|x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1}) \leq \|x\|_p \|w\|_q + \|y\|_p \|w\|_q.
\]

where the last inequality is the Hölder inequality. We have a formula for \( \|w\|_q \), which we use to deduce

\[
\|x + y\|_p^p \leq \|x\|_p \|w\|_q + \|y\|_p \|w\|_q = (\|x\|_p + \|y\|_p) \|x + y\|_p^{p/q}.
\]

Dividing through gives

\[
\|x + y\|_p \leq \|x\|_p + \|y\|_p,
\]

which is what we wanted.

Exercise 1.4. Show that for a given vector \( x \in \mathbb{R}^n \), the function \( p \rightarrow \|x\|_p \) is decreasing on \( p \in [1, \infty) \).

Show further that \( \lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty \).

Proposition 1.5. Given \( x \in \mathbb{R}^n \) and any \( p \in [0, \infty] \)

\[
\|x\|_1 \geq \|x\|_p \geq \|x\|_\infty \geq \frac{1}{n} \|x\|_1.
\]

This follows from Exercise 1.4 and the observation that \( \|x\|_1 \leq n \|x\|_\infty \).
Proposition 1.6. Given $x \in \mathbb{R}^n$ and any $[p,q] \subset [0,\infty]$, there exist constants $c, C > 0$, such that for any $x \in \mathbb{R}^n$, we have

$$C\|x\|_p \geq \|x\|_q \geq c\|x\|_p.$$ 

This follows immediately from 1.5. It may be worthwhile to find the best possible constants $c, C$, but we will not do that here.