Appendix A

Category Theory

1 Categories, Functors and Natural Transformations

We generally disregard problems of size, viz. whether or not something is a set or not.

Definition 1.1. A category \( C \) consists of a collection of objects, \( \text{ob} \, C \) and a collection of morphisms \( C \), such that

1. Every morphism has a source in \( \text{ob} \, C \) and a target in \( \text{ob} \, C \). A morphism \( f \) is often written \( f : X \rightarrow Y \) or \( X \xrightarrow{f} Y \), where \( X \) is the source and \( Y \) is the target.

2. For any two objects \( X \) and \( Y \), there is a set \( C(X, Y) \) or \( C(X, Y) \), consisting of precisely those morphisms of \( C \) having source \( X \) and target \( Y \).

3. For any three objects \( X, Y, Z \) of \( C \), there is a composition of morphisms

\[
\circ : C(X, Y) \times C(Y, Z) \rightarrow C(X, Z)
\]

and this composition is associative in that \( f \circ (g \circ h) = (f \circ g) \circ h \) whenever these composites are defined.

4. For each object \( X \) of \( C \), there exists an identity morphism \( \text{id}_X \in C(X, X) \) such that \( f \circ \text{id}_X = f \) and \( \text{id}_X \circ g = g \) whenever these composites are defined.

Remark 1.2. An easy and standard argument proves that \( \text{id}_X \) is the unique morphism \( X \rightarrow X \) with the stated property.

Notation 1.3. There are categories \( \text{Set}, \text{Gr}, \text{Ab} \), of sets, groups, abelian groups, and many other similar categories of objects commonly studied in mathematics. These are generally large categories, in that the collection of objects does not form a set.

Example 1.4. There are also small categories, where the collection of objects forms a set, and therefore the collection of morphisms also forms a set (under our hypotheses). For instance, given any partially ordered set \( S \), one can construct a category, also called \( S \), where one regards ‘element of’ and ‘object of’ as synonymous, and then declares that \( S(a, b) = \emptyset \) if \( b < a \) and that \( S(a, b) \) consists of one morphism if \( a \leq b \).
3. a bimorphism if it is both a monomorphism and an epimorphism.

**Definition 1.13.** If C is a category, and f : X → Y is a morphism in this category, then we say that f is
1. a split monomorphism if there exists a morphism g : Y → X such that g ∘ f = id_X.
2. a split epimorphism if there exists a morphism g : Y → X such that f ∘ g = id_Y.

**Exercises**
1. Suppose f : X → Y is an isomorphism. Prove that f⁻¹ is uniquely determined by f.
2. Prove that the class of isomorphisms in a category has the two-out-of-three property, namely: if

   \[
   \begin{array}{ccc}
   A & \xrightarrow{f} & B \\
   \downarrow{g} & & \downarrow{h} \\
   \end{array}
   \]

   are composable morphisms such that two of f, g and g ∘ f are isomorphisms, then so too is the third.
3. Prove that the class of isomorphisms in a category has the two-out-of-six property, namely: if

   \[
   \begin{array}{ccc}
   A & \xrightarrow{f} & B \\
   \downarrow{g} & & \downarrow{h} \\
   C & \xrightarrow{h} & F \\
   \end{array}
   \]

   are composable morphisms such that g ∘ f and h ∘ g are isomorphisms, then so too are f, g, h and h ∘ g ∘ f.
4. Determine the monomorphisms, epimorphisms and bimorphisms in the category of sets.
5. Give an example in Top of a bimorphism that is not an isomorphism.
6. Let Haus denote the full subcategory of Hausdorff topological spaces. Give an example in Haus of an epimorphism that is not surjective.

### 2 Functors and Natural Transformations

**Definition 2.1.** Given two categories C and D, a (covariant) functor F : C → D consists of an assignment

\[ F : \text{ob} C \to \text{ob} D \]

and for every pair of objects X, Y in \text{ob} C, a function

\[ F : C(X, Y) \to D(F(X), F(Y)) \]

such that
1. \( F(\text{id}_X) = \text{id}_{F(X)} \) for all object X of C and
2. \( F(f ∘ g) = F(f) ∘ F(g) \) wherever \( f ∘ g \) is defined.

**Example 2.2.** Given any category C, there is an identity functor \( \text{id}_C \).
**Definition 2.3.** Given two categories $\mathbf{C}$ and $\mathbf{D}$, a contravariant functor $F : \mathbf{C} \to \mathbf{D}$ consists of an assignment

$$F : \text{ob} \mathbf{C} \to \text{ob} \mathbf{D}$$

and for every pair of objects $X, Y$ in $\text{ob} \mathbf{C}$, a function

$$F : \mathbf{C}(X, Y) \to \mathbf{D}(F(Y), F(X))$$

such that

1. $F(\text{id}_X) = \text{id}_{F(X)}$ for all object $X$ of $\mathbf{C}$ and
2. $F(f \circ g) = F(g) \circ F(f)$ wherever $f \circ g$ is defined.

**Remark 2.4.** Warning: contravariant functors reverse the direction of morphisms. Failure to keep adequate track of the variance of functors is the category-theoretical analogue of a sign error in arithmetic. These errors are minor, frustrating and common.

**Notation 2.5.** Given a category $\mathbf{C}$, there is an opposite category $\mathbf{C}^{\text{op}}$ having the same collection of objects, but where

$$\mathbf{C}^{\text{op}}(X, Y) = \mathbf{C}(Y, X).$$

One may view a contravariant functor $F : \mathbf{C} \to \mathbf{D}$ as a covariant functor $F : \mathbf{C}^{\text{op}} \to \mathbf{D}$.

**Example 2.6.** There are many functors in mathematics that consist largely of forgetting structures. Such functors are often called “forgetful”, but it is difficult to give a precise definition of what this means. Common examples include:

1. $\mathbb{V} : \bullet \to \to$, forgetting the basepoint.
2. $\mathbb{V} : \to \to \textbf{Set}$, forgetting the topology.
3. $\mathbb{V} : \textbf{Ab} \to \textbf{Grp}$, forgetting that the group is abelian.

**Example 2.7.** There is a canonical functor $\eta : \mathbf{C}^{\text{op}} \times \mathbf{C} \to \textbf{Set}$ given by $\eta(X, Y) = \mathbf{C}(X, Y)$. Fixing either $X$ or $Y$ gives rise to functors

1. $\eta_X : \mathbf{C} \to \textbf{Set}$,
2. $\eta_Y : \mathbf{C}^{\text{op}} \to \textbf{Set}$.

**Definition 2.8.** Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. We say $F$ is

1. full if, for any two objects $X, Y$ of $\mathbf{C}$, the function $F : \mathbf{C}(X, Y) \to \mathbf{D}(F(X), F(Y))$ is surjective.
2. faithful if, for any two objects $X, Y$ of $\mathbf{C}$, the function $F : \mathbf{C}(X, Y) \to \mathbf{D}(F(X), F(Y))$ is injective.
3. essentially surjective if, for any object $Z$ of $\mathbf{D}$, one can find an object $X$ of $\mathbf{C}$ such that there exists an isomorphism $Z \to F(X)$. 
The unit is formed by letting $\eta_X : X \to \R(L(X))$ be the element of $\C(X, R(L(X)))$ corresponding to $\id_{L(X)} \in D(L(X), L(X))$ under the natural isomorphism of the adjunction. The counit is formed similarly.

**Remark 3.6.** We continue with the notation of the previous definition. The unit and counit have certain universal properties. In the case of the unit, suppose that there is a morphism $f : X \to R(Y)$ in $\C$. Since $L$ and $R$ are adjoint, the morphism $f$ is equivalent to a unique morphism $g : L(X) \to Y$. This morphism can be written, tautologically, as $\id_{L(X)} \circ g : L(X) \to L(X) \to Y$, which, by adjunction, is equivalent to a factorization $f = R(g) \circ \epsilon_X : X \to R(L(X)) \to R(Y)$.

Dually, any morphism $h : L(X) \to Y$ factors uniquely as $\eta_Y \circ L(i) : L(X) \to L(R(Y)) \to Y$.

**Remark 3.7.** If $L : \C \to \D$ and $M : \D \to \E$ are two functors, each left adjoint to functors $R$ and $S$ respectively, then $M \circ L$ is left adjoint to $R \circ S$.

**Proposition 3.8.** Suppose $L, L' : \C \to \D$ are two naturally isomorphic functors and $R, R'$ are right adjoints to $L$ and $L'$. Then $R$ and $R'$ are naturally isomorphic.

This result applies in particular in the case where $L = L'$.

## 4 Diagrams, Limits and Colimits

**Notation 4.1.** If $\I$ is a small category and $\C$ is a category, then a functor $D : \I \to \C$ may be called a diagram. If, for any morphism $f : i \to j$ in the category $\I$, the morphism $D(f)$ depends only on $i$ and $j$, then we say the diagram is commutative.

**Example 4.2.** Not all commonly occurring diagrams are commutative. For instance, pairs of parallel morphisms $X \Rightarrow Y$ appear often but form a commutative diagram only when the two morphisms agree.

**Definition 4.3.** Given a small category $\I$ and a category $\C$, one can define a category $\Fun(\I, \C)$ of $\I$-shaped diagrams. The objects are the functors $D : \I \to \C$, and the morphisms are the natural transformations between them.

**Definition 4.4.** Given a small category $\I$, a category $\C$ and an object $X$ of $\C$, we can form the constant $\I$-shaped diagram with value $X$ by const$_\I(X) : \I \to \C$ by sending all objects to $X$ and all morphisms to $\id_X$.

**Definition 4.5.** Let $\I$ be a small category and $\C$ a category.

Given an $\I$-shaped diagram $D$ in $\C$, a limit of $D$ is an object $\lim D$ of $\C$ and a natural transformation $\Phi : \const_\I(\lim D) \to D$ such that for any object $X$ of $\C$ equipped with a natural transformation $\Psi : \const_\I(X) \to D$, there is a unique map $u : X \to \lim D$ such that $\Psi = \Phi \circ \const(u)$.

Dually, a colimit of an $\I$-shaped diagram $D$ is an object $\colim D$ of $\C$ and a natural transformation $\Phi : D \to \const_\I(\colim D)$ such that for any object $X$ of $\C$ equipped with a natural transformation $\Psi : D \to \const_\I(X)$, there is a unique map $u : \colim D \to X$ such that $\Psi = \const(u) \circ \Phi$.

**Remark 4.6.** Strictly speaking a limit or colimit of a diagram encompasses both the object and the natural transformation of functors—which is to say, the morphisms. In practice, one often refers to the object as the limit or colimit, leaving the morphisms implicit.
Remark 4.7. It follows easily from a standard argument that if \( L \) and \( L' \) are two limits of the same diagram \( D : I \rightarrow C \), then there is a unique isomorphism \( f : L \rightarrow L' \) in \( C \) such that the diagram

\[
\begin{array}{ccc}
\text{const}_I L & \xrightarrow{\text{const}_I(f)} & \text{const}_I L' \\
\downarrow & & \downarrow \\
D & & \\
\end{array}
\]

commutes. A dual statement applies to colimits.

Since they are unique up to unique isomorphism, one often abuses terminology and speaks of “the limit” or “the colimit” of a diagram.

Remark 4.8. There is another view of limits and colimits that is sometimes useful. Suppose the functor \( \text{const}_I \) has a right adjoint \( \ell \). Then a limit of \( D \) is given by the object \( \ell(D) \) and the counit map \( \text{const}_I \ell(D) \rightarrow D \).

Dually, if \( \text{const}_I \) has a right adjoint \( \text{colim} \), the colimit of \( D \) is the unit map \( D \rightarrow \text{const}_I \text{colim}(D) \).

Example 4.9. The language used above is technical. In practice, the idea is simple. Let us consider as a category \( I \) the standard cospan

\[
\begin{array}{ccc}
& \bullet & \\
\downarrow & \nwarrow & \downarrow \\
\bullet & & \bullet
\end{array}
\]

Let \( C = \text{Top} \) be the category of topological spaces. Then the data of an \( I \)-shaped diagram \( D \) consists of three spaces and two continuous functions \( X \rightarrow Y \rightarrow Z \).

The constant-diagram functor takes a space \( W \) and produces \( W \rightarrow W \rightarrow W \), where the morphisms are identities. A natural transformation \( \text{const}(W) \rightarrow D \) is the data of continuous functions \( f : W \rightarrow X \), \( g : W \rightarrow Y \) and \( h : W \rightarrow Z \) such that

\[
\begin{array}{ccc}
W & \xrightarrow{f} & W \\
\downarrow & & \downarrow \\
X & & Y \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & Z \\
\end{array}
\]

commutes, or, more succinctly

\[
\begin{array}{ccc}
W & \xrightarrow{f} & X \\
\downarrow & \nwarrow & \downarrow \\
Y & \xrightarrow{g} & Z
\end{array}
\]

(A.1)

Note further that the dotted arrow is determined by either \( f \) or \( h \), and may be omitted.

The space \( \text{lim} \, D \) and the natural transformation amounts to an object and morphisms fitting in the following diagram

\[
\begin{array}{ccc}
\text{lim} \, D & \xrightarrow{h} & X \\
\downarrow & \nwarrow & \downarrow \\
Y & \xrightarrow{f} & Z
\end{array}
\]

(A.2)
This diagram has the property that if $W$ is as in Diagram (A.1), then there exists a unique map $W \to \lim D$ such that Diagram (A.3) commutes.

$$W \xrightarrow{h} \lim D \xrightarrow{g} X \xleftarrow{f} \lim D \xrightarrow{\varepsilon} X$$

This particular kind of limit is called a fibre product and is written $X \times_Y Z$. While our definition specifies the limit only up to unique isomorphism, we can easily construct an explicit model for $X \times_Y Z$ in the category of topological spaces. Most usually, let $X \times_Y Z$ consist of the subset of pairs $(x, z) \in X \times Z$ such that the image of $x$ and of $z$ in $Y$ agree. Then endow $X \times_Y Z$ with the coarsest topology (fewest open sets) such that the evident projection maps $X \times_Y Z \to X$ and $X \times_Y Z \to Z$ are both continuous.

It is instructive to consider $X \times_Y Z$ in the following cases:

1. When $Y$ is a singleton space.
2. When $X \to Y$ is the inclusion of a subspace.

**Remark 4.10.** By uniqueness of adjoints and of unit or counit maps, if a limit or colimit of a diagram exists, it is unique up to unique isomorphism.

**Notation 4.11.** A category in which all limits can be constructed is complete and one in which all colimits can be constructed is cocomplete. The following categories are all complete and cocomplete:

1. **Set**.
2. **Top** and **Top**$^\times$.
3. $R$-**Mod**.

The full subcategory **Haus** of Hausdorff topological spaces is complete but not cocomplete.

**Notation 4.12.** If $D$ is a diagram in $C$ consisting of a family of objects $\{X_i\}_{i \in I}$ and no nonidentity arrows, then a limit of $D$ is called a product of $\{X_i\}_{i \in I}$ and a colimit of $D$ is called a coproduct of $\{X_i\}_{i \in I}$. The product of topological spaces is an example of a categorical product, and the disjoint union of topological spaces is an example of a categorical coproduct.

**Notation 4.13.** If $D$ is a diagram in $C$ of the form

$$A \xrightarrow{f} B \xrightarrow{g} C$$

then a limit of $D$ is called a pullback of $D$, and often denoted $A \times_C B$. 


The dual concept is the \textit{pushout}, a colimit of.

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & C
\end{array}
\]

\textbf{Proposition 4.14.} Suppose $F : C \rightarrow C$ is a functor between complete categories such that $F$ has a left adjoint, $L$. Suppose further that $D$ is a diagram in $C$. Let $\lim D$ be a limit of $D$. Then $F(\lim D)$ is a limit of $F(D)$.

Dually, suppose $F : C \rightarrow C$ is a functor between cocomplete categories such that $F$ has a right adjoint, $R$. Suppose further that $D$ is a diagram in $C$. Let $\operatorname{colim} D$ be a limit of $D$. Then $F(\operatorname{colim} D)$ is a colimit of $F(D)$.

\textbf{Remark 4.15.} Let $C$ be a category. Consider the empty diagram $D$. If $\lim D$ exists, then it is an object $\ast$ such that all objects $X$ of $C$ are equipped with a unique morphism $X \rightarrow \ast$. Such an object $\ast$ is called a \textit{terminal} object of $C$. Any two terminal objects are isomorphic by a unique isomorphism.

Dually, the colimit of an empty diagram is called an \textit{initial} object; such an object may often be denoted $\varnothing$. If an object is both initial and terminal, then it is called a \textit{zero object}.

\textbf{Exercises}

1. The forgetful functor $V : \text{Ab} \rightarrow \text{Set}$ has a left adjoint, $L$. Describe the unit map $\epsilon : S \rightarrow V(L(S))$.

2. Show that $V : \text{Ab} \rightarrow \text{Set}$ does not preserve colimits. For instance, consider the colimit of a diagram consisting of two nonzero abelian groups and no nontrivial arrows. Therefore $V$ does not have a right adjoint.

3. Let $R$ be a ring and let $\textbf{M}$ denote the category of $R$-modules and $R$-linear maps, and let $f : M \rightarrow N$ be a morphism in $\textbf{M}$. Describe the limit of the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
M & \longrightarrow & N.
\end{array}
\]

Express the cokernel of $f$ as the colimit of a diagram.

4. Consider the forgetful functor $V : \text{Top} \rightarrow \text{Top}$. Describe a left adjoint to this functor. Prove that $V$ does not have a right adjoint.

5. Let $X$ be a locally compact Hausdorff space, and consider the adjunction between $\times X$ and $\mathcal{C}(X, \cdot)$ in $\text{Top}$. Describe the counit of this adjunction.