Def 11.1 Let $\mathbb{R}, \mathbb{C}$ be topological spaces and $p : \mathbb{C} \to \mathbb{R}$ a continuous map. We say $p$ is a covering space map if $\forall x \in \mathbb{R}$, there exists $U, x$ open s.t. $p^{-1}(U) = \bigsqcup_{i \in I} V_i$, s.t. $p \big|_{V_i} : V_i \to U_i$.

Example 11.2 1. Any identity map is a covering map.
2. $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is covering. Indeed, we already have seen that for any $x \in S^1$, there exists some $V \in \mathbb{R}$ s.t. $p \big|_{V} : V \to p(V) \times \{x\}$ is a homeomorphism. Then it suffices to observe that $p^{-1}(p(V)) = \{x + 2\pi n | n \in \mathbb{Z}, v \in V\}$ is a disjoint union.

3. Let $S' \xrightarrow{f^0} S$ be defined as follows: view $S' = \{z \in \mathbb{C} | |z| = 1\}$

$$f^0 : S' \to S' \quad f^0(z) = e^z.$$ i.e. $f^0(\cos \theta + i \sin \theta) = \cos(n\theta) + i \sin(n\theta)$.

$f^0$ is a covering map.
(Exercise).

**Def 11.3.1** If \( p : C \to \tilde{X} \) is a map of spaces (e.g. a covering space map) and \( f : Y \to \tilde{X} \) an \( \tilde{X} \)-map:

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & \tilde{X} \\
\downarrow & & \downarrow \\
\quad & & \\
\quad & & \\
\end{array}
\]

then a map \( \tilde{f} : Y \to C \) s.t. \( p \circ \tilde{f} = f \) is called a \textit{lift} or \textit{lifting} of \( f \) to \( C \).

**Def 11.4.1** If \( \{ \tilde{X}_\alpha \}_{\alpha \in \mathcal{A}} \) is a family of disjoint top spaces, then \( \coprod_{\alpha \in \mathcal{A}} \tilde{X}_\alpha \), their disjoint union is the \( U \tilde{X}_\alpha \) given the finest topology making all inclusions \( \tilde{X}_\alpha \to \coprod_{\alpha \in \mathcal{A}} \tilde{X}_\alpha \) continuous.

i.e. \( U \subseteq \coprod_{\alpha \in \mathcal{A}} \tilde{X}_\alpha \) is open if

\[
U \subseteq \tilde{X}_\alpha \subseteq \coprod_{\alpha \in \mathcal{A}} \tilde{X}_\alpha \quad \text{& this generates a topology.}
\]
Note: $\tilde{\xi}_0 \rightarrow N \tilde{\xi}_0$ in homomorphic to its image.

Prop 11.51 (Unique lifting property)

Let $p: C \rightarrow \tilde{X}$ be a covering space and let $Y$ be a connected space. Suppose $\tilde{f}_1, \tilde{f}_2$ are two lifts that agree at one point of $Y$. Then $\tilde{f}_1 = \tilde{f}_2$.

Proof: We show the set of points where two lifts agree is both open and closed, and since it is not empty, it must be $Y$.

Suppose $y \in Y$ is s.t.

then $\exists U \ni f_i(y)$ minimizing the cover

Restrict attention to

\[
\begin{array}{c}
\tilde{f}_i \quad \overset{N}{\rightarrow} \\
\bigcap_{i \in \mathbb{Z}} U_i \quad f_i \\
\end{array}
\]

Then $y \in f^{-1}(U)$ $\rightarrow$ $U$
If \( \tilde{f}_1(y) \neq \tilde{f}_2(y) \), then \( \tilde{f}_1(y) \in U_i \neq U_j \).

\( \tilde{f}_1^{-1}(U_i) \cap \tilde{f}_2^{-1}(U_j) \) gives an open set on which \( \tilde{f}_1, \tilde{f}_2 \) differ.

Conversely, if \( \tilde{f}_1^{-1}(y) \cap \tilde{f}_2^{-1}(y) \in U_i \), then \( \tilde{f}_1^{-1}(U_i) \cap \tilde{f}_2^{-1}(U_i) \) gives an open set where they agree. \( \square \)

**Prop 11.61 (The Homotopy Lifting Property)**

Given a covering space \( p: C \to \tilde{X} \) and a commuting diagram

\[
\begin{array}{ccc}
Y \times I & \xrightarrow{h} & Y \\
\downarrow p & & \downarrow p \\
Y \times I & \xrightarrow{\tilde{f}} & C \\
\end{array}
\]

there exists a unique lift \( \tilde{g}: Y \times I \to C \) such that both triangles commute, i.e.

a) \( \tilde{g}(y, 0) = \tilde{f}(y) \)

b) \( p(\tilde{g}(y, t)) = h(y, t) \).

**Proof:** Any lift must be unique - m
fact, if $y \in y$, then $y \cup I$ is connected if 11.6 applies to show $A \mid y \times I$ is unique if $\exists I$ exists.
Let
\[ U = \{ u \in X \mid u \text{ open}, p^{-1}(u) = \prod_{i \in I} u_i, p|_{u_i} \approx u_i \} \]
\[ \forall i \]

Consider \( H^i(U) \); this covers \( Y \times I \).

Fix \( y \in Y \). Then \( y \times I \subseteq Y \times I \) is compact and so is covered by finitely many sets of the form \( H^i(U) \).

\[ H^i(U_1) \cup H^i(U_2) \cup \ldots \cup H^i(U_n) \]

For each \( t \in I \), choose a \( u_i \) so that \( (t - \varepsilon, t + \varepsilon) \cap i \subseteq \{ y \} \times (t - \varepsilon, t + \varepsilon) \)

for some \( i \).

We can again restrict to finitely many such \( t \) giving a cover of \( I \) by \( u_i \) \( s.t. \)

\[ y \times I \subseteq y \times I_i \subseteq H^i(U_i) \]

get them use the tube lemma to find
\( w_1, w_2 \subseteq Y \) open s.t.
\[
\chi_{w_1} \cap \chi_{w_2} = \emptyset.
\]
\( \chi_{w_1} \times \chi_{w_2} \subseteq \chi_{w_1}'(U, \cdot) \)

& intersecting gives an open \( W \in Y \)
\[
W \times \chi_{w_1} \subseteq \chi_{w_1}'(U, \cdot)
\]

& \( W \times \chi_{w_1} \cap \ldots \cap W \times \chi_{w_k} \subseteq \chi_{w_1} \times \ldots \times \chi_{w_k} \).

We can easily construct a lift \( C^w : \chi_{w_1} \to C \)

Do this for all \( y \in Y \)

then define

\[
G(y, t) = G^w(y, t) \quad \text{where } y \in w
\]

by uniqueness, the choice of \( w \) doesn't affect \( G^w(y, t) \). Moreover, if \( U \) is open \& \( C \)

then \( (y, t) \in G^w(U) \iff (y, t) \in (C^w)^{-1}(U) \)

for some \( w \) \& \( (C^w)^{-1}(U) \) is open

\( (C^w)^{-1}(U) \subseteq C^{-1}(U) \), \( \forall U \).
Prop 11.7.1 Let $p : C \to \tilde{X}$ be a covering space. Let $c \in C$, then the map

$$\pi_1(C, c) \xrightarrow{p_*} \pi_1(\tilde{X}, p(c))$$

is injective & the image of $p_*$ consists of homotopy classes of loops $\gamma : I \to \tilde{X}$ that lift to loops $\tilde{\gamma} : I \to C$ based at $c$.

Proof:

Consider $\delta : I \to C$ in ker $p_*$, then $p \circ \delta$ is a loop in $\tilde{X}$ that is homotopic to the constant loop $\varepsilon_{p(c)}$.

We can lift the contracting homotopy of $p \circ \delta$ to get a contracting homotopy of $\delta$ using uniqueness.
The second claim is easy.

**Prop 11.8** Let \( p : \tilde{X} \to X \) be a covering space, let \( x_0 \in \tilde{X} \) and write \( \tilde{x} = p^* (x_0) \). For any \( \gamma : (S^1, \underline{s}) \to (X, x_0) \) and \( c \in I \), write \( l(\gamma, c) \) for the lift of \( \gamma \) starting at \( c \). Suppose \( \gamma, \delta \) are loops in \( \tilde{X} \) at \( x_0 \) and \( l(\gamma, c) = c' \):

1. \( l(\gamma \circ \delta, c) = l(\gamma, c) \ast l(\delta, c') \)
2. \( l(\gamma, c) = l(\gamma, c') \)
3. \( [l(\gamma, c)] \) depends only on \([\gamma], c\)

**Prop 11.9.** In the notation of the previous prop., given \( \gamma, c, c' \), the point \( c' = l(\gamma, c)(1) \) is determined by \( c' \) if the class of \([\gamma]\) in \( \pi_1(X, x_0) \) is \([\gamma'] \) with \( \gamma'(1) = c \). Therefore, \( l(\gamma, c)(1) = l(\gamma', c)(1) \).

**Def 11.10.** Topological group.
A top group is \((G, \ast, e, inv)\).
A top group action is if \( G \) is discrete.
An action is free if \( e \times X \to X \times X \) is inj. and has if
Corollary 11.11: Suppose \( X \) is a topological space, \( p: C \to X \) a covering space, and \( C \) is path connected. Let \( c_0 \in C \) be a basepoint and \( x_0 = p(c_0) \). Write \( f = \tilde{p}(x_0) \) (so \( c_0 \in \Gamma \)) then there is a bijection

\[
\mathcal{F} \to \pi_1(C, c_0) \setminus \pi_1((x_0, \mathfrak{g}) = \varnothing
\]

given by assigning to \( f \in \mathcal{F} \) the class \( \pi_1(C, c_0)[\gamma] \).

Proof: Any path \( \gamma \) in \( C \) from \( c_0 \) to \( f \) gives an element of \( \pi_1((x_0, \mathfrak{g}) \) by \( p_*(\gamma) \).

If two paths \( \gamma, \gamma' \) go from \( c_0 \) to \( f \) then \([\gamma'] \cdot [\gamma]^{-1} \) is in \( \pi_1(C, c_0) \)

so \([\gamma'] \cdot [\gamma]^{-1} \in \pi_1(C, c_0) \)

so the class of \( p_*(\gamma) \) in the quotient \( \pi_1(C) \setminus \pi_1((x_0, \mathfrak{g}) \)

depends only on the endpoint, \( f \).

Therefore \( b \) is well defined.

To see it is injective, suppose \( b(f_0) = b(f_1) \)

find paths \( \gamma_0, \gamma_1 \) starting at \( c_0 \) and ending at \( f_0, f_1 \)

then \([\mathfrak{g}] \in \pi_1(C, c_0) \) s.t. \( p_*(\mathfrak{g}) = p_*(\gamma_0) \cdot p_*(\gamma_1) \)

\( \mathfrak{g} \) a loop in \( C \). Then by lifting homotopies

\( s \times \gamma_0 \sim \gamma_1 \) and endpoints

\( \Rightarrow \) they end at the same point \( \Rightarrow f_0 = f_1 \).
Surjectivity is easier (given by lifting paths) if

\textbf{Def 11.12} If \( p : C \rightarrow X \) is a surjective covering space map where \( C \) is simply connected, then \( C \) is a \textit{universal cover} of \( X \).

\textbf{Cor 11.13} Given a universal cover \( p : C \rightarrow X \) and any \( x_0 \in X \), \( \pi_1(X, x_0) \cong p^{-1}(x_0) \), the bijection being of principal homogeneous \( \pi_1(X, x_0) \)-spaces. In fact the bijection is of groups if we think more.

\textbf{Example 11.14} \( p : IR \rightarrow S' \) given by
\[ p(\theta) = (\cos 2\pi \theta, \sin 2\pi \theta) = e^{i2\pi \theta}. \]
is a universal cover.

There is a bijection \( \pi_1(S', (1,0)) \cong p^{-1}(1,0) \) of \( \{ 0, \pm 1, \pm 2, \ldots \} \times IR \).

Any loop \( \gamma : (S', (1,0)) \rightarrow (S', (1,0)) \) also \( \gamma : (I, 0) \rightarrow (S', (1,0)) \) \( \gamma(0) = (1,0) \)
is homotopic (rel endpoints) to one of
\[ l_n(\theta) = (\cos 2\pi n \theta, \sin 2\pi n \theta), \ n \in \mathbb{Z}. \]

For a given path \( \gamma : I \rightarrow S' \), the integer \( n \) is called the \textit{winding number} of \( \gamma \), denote \( [\gamma] = [l_n] \).
One checks directly that \( l_n \times l_m \cong l_{n+m} \) \((n, m \geq 0)\) and \( l_1 \times l_{-1} \cong l_0 \Rightarrow l_a \times l_b \cong l_{a+b} \) \(a, b \in \mathbb{Z}^2\).

The van Kampen Theorem

Def 11.15: Suppose \( \mathfrak{X} \) and \( \mathfrak{Y} \) are two groups, then by the free product of \( \mathfrak{X} \) and \( \mathfrak{Y} \), we mean the group \( \mathfrak{X} \ast \mathfrak{Y} \) consisting of words of length 20 with entries in \( \mathfrak{X} \) and \( \mathfrak{Y} \) subject to the relations:

\[
wx_1 x_2 w' \sim w(x_1 x_2)w' \quad x_1, x_2 \in \mathfrak{X} \\
w e_{\mathcal{X}} w' \sim w w' \quad w e_{\mathcal{Y}} w' \sim w w' \\
w y_1 y_2 w' \sim w(y_1 y_2)w' \quad y_1, y_2 \in \mathfrak{Y}
\]

the group operation on \( \mathfrak{X} \ast \mathfrak{Y} \) is given by concatenation.

Remarcs: 1) The free product of \( \mathfrak{X} \ast \mathfrak{Y} \) is the group generated by combining elements from \( \mathfrak{X} \) with those of \( \mathfrak{Y} \) & adding “no further relations.”

2) It is possible to define \( \mathfrak{X} \ast \bigwedge_{i \in I} A_i \) as well.

3) \( \text{Hom} \left( \mathfrak{X} \ast \mathfrak{Y}, G \right) \cong \text{Hom}(\mathfrak{X}, G) \times \text{Hom}(\mathfrak{Y}, G) \) (this is a rank to a definition of \( \mathfrak{X} \ast \mathfrak{Y} \) via universal property).
Def 11.16.1 Let $\bar{X}, \bar{Y}$ be groups and $\phi: \bar{X} \to \bar{Y}$ a homomorphism. The **amalgamated product** of $\bar{X}$ with $\bar{Y}$ over $\bar{Z}, \bar{X} \times_{\bar{Z}} \bar{Y}$ is the group given as

$$(\bar{X} \times \bar{Y}) / N$$

where $N$ is the normal subgroup generated by elements $\phi(z) \psi(z)^{-1}$.

Remarks: *) This is the group generated by elements of $\bar{X}, \bar{Y}$ subject only to the relations $\phi(z) = \psi(z)$.

*) This satisfies the universal property

$$\text{Hom}(\bar{X} \times_{\bar{Z}} \bar{Y}, A) \cong \text{Hom}(\bar{X}, A) \times \text{Hom}(\bar{Y}, A).$$

**C weave pushout presentation**

$$\begin{array}{ccc}
\bar{X} & \to & \bar{Y} \\
\downarrow & & \downarrow \\
\bar{Z} & \to & \bar{Y}
\end{array}$$

Prop 11.17 (The Seifert-van Kampen theorem).

Let $\bar{X}$ be a topological space s.t.

$$\bar{X} = U_1 \cup U_2$$

where $U_1, U_2$ are open and $U_1 \cap U_2$, $U_1, U_2$ are path connected. Let $x_0 \in U_1 \cap U_2$, then

$$\pi_1(\bar{X}, x_0) \cong \pi_1(U_1, x_0) \times \pi_1(U_2, x_0) / \pi_1(U_1 \cap U_2, x_0).$$
Example 11.18 | Suppose \( \overline{S} = U \cup U_2 \) (\( U, U_2 \) open)

where \( U, U_2 \) are simply connected. If \( U \cap U_2 \) path connected (nonempty), then \( \overline{S} \) is simply connected.

Proof: \( \overline{S} \) is path connected as the union of two path connected spaces with nonempty intersection.

Then apply the Brouwer theorem with \( x_0 \in U \cap U_2 \) then \( \pi_1(\overline{S}, x_0) = \{e\} \).

Note that if \( n \geq 2 \), then \( S^n \) is the one-point compactification of \( R^n \). Let \( n \) be
the north pole \((0, -1, 0, 1)\) & \( S^n \) the south pole \((0, 1, 0, -1)\).

Then
\[
S^n \setminus \{n\} \cong R^n
\]
\[
S^n \setminus \{n, s\} \cong R^n
\]
\[
S^n \setminus \{n, s\} \cong R^n \setminus \{0\}
\]
which is path connected (if \( n \geq 2 \) — this fails if \( n = 1 \)).

Therefore, \( \pi_1(S^n, s) = \{e\} \) & \( S^n, n \geq 2 \)
is simply connected.

Example 11.19 | Let \( (X, x_0), (Y, y_0) \) be two spaces
s.t. \( X \) open sets \( U \ni x_0 \) \& \( Y \ni y_0 \) s.t. \( U \)
deformation retracts onto \( Sx_0 \) \& \( V \) def. retracts onto \( Sy_0 \).
(This is common — open subsets of \( R^n \) have this)
property.)

Let \( \mathbb{X} \lor Y \) denote the space \( \mathbb{X} \lor Y / \sim \)
where \( \sim \) is the relation \( x_0 \sim y_0 \)
given the basepoint \((x_0, y_0)\).

Then \( \mathbb{X} \lor Y = (U \lor V) \lor (X \lor V) \)
\(U \lor Y \sim Y, \mathbb{X} \lor V \sim \mathbb{X}, \quad (U \lor Y) \cap (X \lor V) = U \lor V \).

So \( \pi_1(\mathbb{X} \lor Y, \text{basepoint}) \cong \pi_1(\mathbb{X}, x_0) \times \pi_1(Y, y_0) \).

In particular: \( S' \lor S' \) has \( \pi_1(S' \lor S', s) \cong \mathbb{F}_2 \times \mathbb{F}_2 \).

This group is generated by
two letters \( x, y \) with no
relations beyond the obvious \( x \cdot x^{-1} = e = y \cdot y^{-1} \).

In particular \( x \cdot y = y \cdot x \); \( \mathbb{F}_2 \) is not abelian.

We define \( \mathbb{F}_1 = \mathbb{Z} \)
\( \mathbb{F}_n = \pi_1(S' \lor \cdots \lor S', s) \), the free group on
\( n \) letters.

Then \( n \geq 2 \), \( \mathbb{F}_n \) is not abelian.

(If \( G \) is any group that can be generated
by \( n \) elements, then \( G \) a surjective map
\( \mathbb{F}_n \twoheadrightarrow G \).)
Example 11.20

T

Torus minus a point

\[ T \setminus \text{point} \cong S^1 \times S^1 \text{ and the loop} \]
\[ [\mu] = [\alpha] [\beta] [\alpha^{-1}] [\beta^{-1}] \]

\[ T \setminus U \]

\[ \mu \cap N \]

\[ (T \amalg Y) / \mu \cap N \]

has fundamental group given by
Prop 11.22 (The Lebesgue Covering Lemma).

Let \( \mathbb{R} \) be a compact metric space \( \mathbb{R} \) and \( U \) an open cover of \( \mathbb{R} \). There exists some \( \delta > 0 \) such that \( \forall x \in \mathbb{R} \), \( \exists U \in \mathcal{U} \) for some \( U \in \mathcal{U} \).

Proof: We may replace \( U \) by a finite subcover \( \{ U_1, U_2, \ldots, U_n \} \).

For each \( x \in \mathbb{R} \) define

\[
\delta(x) = \sup \{ t \in (0, \infty) \mid B(x, t) \subseteq U \text{ for some } U \in \mathcal{U} \}
\]

We claim \( \delta: \mathbb{R} \to (0, \infty) \) is a continuous function given \( \varepsilon > 0 \). If \( d(x, y) < \varepsilon \), then

\[
B(y, \frac{\varepsilon}{2}) \subseteq B(x, t) \subseteq B(y, t + \varepsilon)
\]

for any \( t \)

\[
\Rightarrow |s(x) - s(y)| < \varepsilon
\]

Therefore, \( s(x) \) attains a minimum value; i.e., \( \exists \delta > 0 \)

such that \( \forall x \in \mathbb{R} \), \( \exists U \in \mathcal{U} \) for some \( U \in \mathcal{U} \).
Def 11.23 | A groupoid $G$ consists of a set of objects $ob G$ and morphisms $mor G$ so that
1. $G$ is a category (small cat)
2. all morphisms are invertible.

Observation 11.24 | A groupoid having only one object is a group.

Def 11.24 | A morphism of groupoids is a functor $\phi : G \to G'$.
$\phi : ob G \to ob G'$
$\phi : mor G \to mor G'$

preserving composition of identities.

Recall the fundamental groupoid of a space $\overline{x}$ has $\text{Ob} \Pi(\overline{x}) = \overline{x}$ (points of $x$)
$\text{Mor}(\overline{x}, \overline{y}) = \text{Hom} \text{closes of paths}$
$\gamma : I \to \overline{x}$
$\gamma(0) = \overline{x}, \gamma(1) = \overline{y}$

$\Pi(\cdot)$ is a functor

Prop 11.25 | (Van Kampen for groupoids) Let $\overline{x}$ be a set. Let $U, V$ be open sets s.t. $U \cup V = \overline{x}$
Let $G$ be a groupoid. Suppose

$$
\begin{array}{c}
\Pi(U \cup V) \\ \downarrow \\
\Pi(U) \\ \downarrow \\
\Pi(U) \\
\end{array} \quad \xrightarrow{\psi} \quad 
\begin{array}{c}
\Pi(U) \\ \downarrow \\
\Pi(U) \\
\downarrow \\
\Pi(U) \\
\end{array}
$$

commutes, the $\Theta$ making diag commute.

Proof: we have to define $\Theta$ for

1. points of $\bar{X}$ (obj of $\Pi \bar{X}$)
2. homotopy classes of paths in $\bar{X}$ (mor)

1. Points is easy. Any point in $\bar{X}$ is either in $U$ or in $V$, & define $\Theta(x) = \phi(x)$ or $\psi(x)$ as appropriate — if it's in both then these agree.

2. Morphisms are harder: For any $\ell \in \ker \Pi(U)$
we define $\Theta[\ell] = \phi[\ell]$
& $\Theta[\ell] = \psi[\ell]$ in the other case $\ell \in \ker \Pi(U)$.
Let \( \gamma \) be a life of paths in \( X \). A partition of \( I \) is a sequence \( t_0 < t_1 < t_2 < t_3 < \ldots < t_{n-1} < t_n = 1 \).

There exists some partition of \( I \) s.t. \( \gamma([t_i, t_{i+1}]) \subseteq U \) or \( \gamma([t_i, t_{i+1}]) \subseteq V \) then take \( \gamma_i : I \to X \) to be \( \gamma |_{[t_i, t_{i+1}]} \) rescaled to take unit time.

Then \( O(\gamma) = O(\gamma_1) \cdot O(\gamma_2) \cdots O(\gamma_n) \)

problemes: what if we use a different partition? First, consider adding a point \( t' \) somewhere - this will lie in an interval \([t_i, t_{i+1}]\) so the resulting \( \gamma([t_i, t']) \times \gamma([t', t_{i+1}]) = \gamma([t_i, t_{i+1}]) \)

\( O(\gamma) \) is not changed if we add finitely many points to the interval. Any two partitions have a common refinement, so \( O(\gamma) \) does not depend on the partition.

Now what if we have a different path \( \delta = \gamma \)? Take a homotopy \( H : I \times I \to X \); using the Lebesgue covering lemma, \( \delta \) the shape we had, we
can divide the square $s_{n,m}$ into smaller squares such that $H(s_{i,j}) \subseteq U$ or $H(s_{i,j}) \subseteq V$.

Now we take each $s_{i,j}$

$$
\epsilon_{i,j} \quad \alpha_{i,j} \quad \epsilon_{i,j+1} \quad \alpha_{i+1,j}
$$

$\epsilon_{0,j}$ is a count path at $\gamma(0)$, $\alpha_{n,j}$ is the path of $\gamma(n)$

$$\Theta(\epsilon_{1,j}) \ast \ldots \ast \Theta(\epsilon_{n,j}) = \Theta(\gamma)$$

and

$$\Theta(\epsilon_{i,j+1}) = \Theta(\alpha_{i,j})^{-1} \ast \Theta(\epsilon_{i,j}) \ast \Theta(\alpha_{i+1,j})$$

$$\Theta(\epsilon_{1,j+1}) \ast \Theta(\epsilon_{2,j+1}) \ast \ldots \ast \Theta(\epsilon_{n,j+1}) =$$

$$\Theta(\epsilon_{1,j}) \ast \Theta(\alpha_{2,j}) \ast \Theta(\alpha_{3,j}) \ast \ldots \ast \Theta(\epsilon_{n,j})$$

so in particular

$$\Theta([\gamma]) = \Theta([\delta])$$

so the map $\Theta$ is well defined.

Proof that $\Theta([\gamma \circ \delta]) = \Theta([\gamma]) \ast \Theta([\delta])$ is
Prop 11.26: Let $\mathcal{X}$ be a path connected space and $x_0 \in \mathcal{X}$ a basepoint. For each point $x \in \mathcal{X}$ choose a path $\alpha_x$ from $x$ to $x_0$. Then the morphism $\pi_1(\mathcal{X}) \rightarrow \pi_1(\mathcal{X}, x_0)$ by sending the path $x \rightarrow x_0$ to $\alpha_x \cdot \gamma \cdot \alpha_x^{-1}$ satisfies $\pi_1(\mathcal{X}, x_0) \xrightarrow{\text{incl.}} \pi_1(\mathcal{X}) \xrightarrow{\imath^*} \pi_1(\mathcal{X}, x_0)$.

Proof of S-vK Theorem:

We can assume $U, V \cong \mathcal{X}$ are path connected (components other than the component of $x_0$ will not appear).

Choose paths from each point in $\mathcal{X}$ to $x_0$, then we obtain a diagram of parallel
Then we construct \( \phi \); if we had two different choices of \( \phi \), then we'd get two loops:
\[
\pi_1(\bar{x}) \to A, \text{ a contradiction.}
\]

So \( \pi_1(\bar{x}) \) is \( \cong \) the amalgamated product
\[
\pi_1(u) *_{\pi_1(u \cap v)} \pi_1(v)
\]
Covering Group Actions

Def 11.28 Let $G$ be a (possibly non-discrete) topological group and $X$ a top. space & $a: G \times X \to X$ an action map. Then define the quotient of $X$ by $G$ as the set of equivalence classes under the relation $x \sim g x$, endowed with the quotient topology: $q: X \to \underline{G}/X$ (or $X/G$).

This is also referred to as the orbit space of the action.

Def 11.27 Suppose $G$, a discrete group, acts continuously on a topological space $X$ via $a: G \times X \to X$. We say the action is a covering action if, for all $x \in X$, there exists an open set $U \ni x$ such that $g U \cap U \neq \phi = g = e$.

This is stronger than being a free action.

Prop 11.28: If $G$ is a discrete group & $a: G \times X \to X$ is a covering space action, then the map $p: X \to \underline{G}/X$ is a covering space, and if $x_0 \in X$, then $p'(x_0) = G$.

Proof: For a given $y \in \underline{G}/X$, find some $x \in X$ s.t. $p(x) = y$ & there exist $U \ni x$ s.t. $U \cap g U = \phi \forall g \in G \setminus \{e\}$
1. \( p_1^n : \mathbb{R} \to \mathbb{R}^n \) is a homomorphism.

2. \( p_1^n(p(n)) = \frac{1}{g(n)} \) for all \( g(n) \).

Example 11.29: \( \mathbb{Z} \subset \mathbb{R} \) with \( (n, r) \mapsto 2n + r \)
gives a quotient \( \mathbb{R} \to \mathbb{R} / \mathbb{Z} \equiv \mathbb{S}^1 \).

b) If \( \mu_2 = \{ \pm 1 \} \) then \( \mu_2 \subset \mathbb{S}^n \) by \( \mu_2 \subset \mathbb{S}^n \) by mult.

\[
\mu_2 \subset \mathbb{S}^n \quad \text{and} \quad \mu_2 \subset \mathbb{S}^n \quad \text{by mult.}
\]

\[
\mu_2 \subset \mathbb{S}^n \quad \text{and} \quad \mu_2 \subset \mathbb{S}^n \quad \text{by mult.}
\]

As \( \mathbb{S}^n \to \mathbb{R}P^n \) is a covering map

\[
(\mathbb{S}^n \cong \mathbb{R}P^n \quad \text{since} \quad \pm 1).
\]