I denote \( g \circ f \).

**Def 10.** Given two maps \( f, g : X \to Y \), a homotopy between them is a map

\[ H : X \times I \to Y \]

such that

\[ H_{|\{0\}} = f \quad H_{|\{1\}} = g. \]

We say \( f, g \) are _homotopic_.

Alternatively,

\[ H : X \to C(I, Y) \]

such that

\[ C(\{0\}, Y) = f \quad C(\{1\}, Y) = g. \]

or if \( X \) is path-connected.
\[ H : I \rightarrow \mathcal{C}(X, Y) \]
\[ H(0) = f, \quad H(1) = g. \]

The maps \( f, g \) are **homotopic**.

we write \( f \simeq g \).

**Prop 10.2**: \( \simeq \) is an equivalence relation on maps \( \overline{X} \rightarrow Y \), i.e.

i) \( f \simeq f \)

ii) \( f \simeq g \Rightarrow g \simeq f \)

iii) \( f \simeq g, \ g \simeq h \Rightarrow f \simeq h \).

**Proof**: i) is easy

ii)

given \( H : \overline{X} \times I \rightarrow Y \) giving \( f \simeq g \),

**define** \( -H : \overline{X} \times I \rightarrow Y \) by

\[-H(x, t) = H(x, 1-t)\]
iii) given $H_1$, from $f$ to $g$, $H_2$, from $g$ to $h$

define $H_1 \times H_2 : \mathbb{X} \times I \rightarrow \mathbb{Y}$ by

$$(H_1 \times H_2)(x, t) = \begin{cases} H_1(x, 2t); & t \leq \frac{1}{2} \\ H_2(x, 2t - 1); & t > \frac{1}{2} \end{cases}$$

Def 10.3: The set of maps $g$ s.t. $g = f$ is called the homotopy class of $f$. We write $\text{Ho}(\mathbb{X}, \mathbb{Y})$ for the set of homotopy classes of maps.
Prop 10.1 If \( f_1, f_2 : \overline{X} \to Y \) and \( g_1, g_2 : Y \to \overline{Z} \) are two pairs of maps and \( f_1 \simeq f_2 \) and \( g_1 \simeq g_2 \), then \( g_1 \circ f_1 \simeq g_2 \circ f_2 \).

Proof: Let \( H_1 \) be a homotopy from \( f_1 \) to \( f_2 \) and \( H_2 \) a homotopy from \( g_1 \) to \( g_2 \) then \( H : \overline{X} \times I \to Z \)

\[
H(x, t) = H_2( H_1( (x, t), t ) )
\]

This yields the required homotopy. \( \square \)

Based spaces

Def 10.5 A **based** or **pointed** space is a space \( \overline{X} \) equipped with a distinguished element \( x_0 \in \overline{X} \), or equivalently a distinguished map \( q \times \overline{X} \to \overline{X} \) to the one-point space.
A map of based spaces $(\tilde{X}, x_0) \rightarrow (Y, y_0)$ is a map $f: \tilde{X} \rightarrow Y$ s.t. $f(x_0) = y_0$.

Equivalently:

```
\[ \begin{array}{c}
\tilde{X} \\
\downarrow f \\
\tilde{X}
\end{array} \quad \Rightarrow \quad
\begin{array}{c}
Y \\
\downarrow f \\
Y
\end{array} \]
```

When we need it to be based, we will have $I$ at $x_0$. So $I = I/\{0\}$ will be also based here.

```
\begin{array}{c}
I \\
\downarrow \\
I
\end{array}
```

**Def 10.6** If $(X, x_0), (Y, y_0)$ are based spaces, a based homotopy of maps $f_1, f_2$ is a map

$H: \tilde{X} \times I \rightarrow Y$

based at $(x_0 \times 0)$

s.t. $H$ is a homotopy from $f_1$ to $f_2$ & $H(x_0, t) = y_0 \quad \forall t$. 

\[ \left( x_0 \times I \right) \]
The set of based homotopy classes is written $[\overline{X}, x_0]$, $(Y, y_0)$ or $[\overline{X}, Y]$ where $x_0, y_0$ are clear.

More generally, if $A \subseteq \overline{X}$ is a subspace and $Y$ a space, and $f_1, f_2 : \overline{X} \to Y$ are s.t. $f_1|_A = f_2|_A$, a homotopy of $f_1$ to $f_2$ relative to $A$ is a map

$$H : \overline{X} \times I \to Y$$

s.t.

1. $H(x, 0) = f_1(x)$
2. $H(x, 1) = f_2(x)$
3. $H(a, t) = f_1(a) = f_2(a)$ \forall $a \in A$.

We write $f_1 \simeq f_2$ (rel $A$).

10.2 & 10.4 apply (rel $A$).
Def 10.7] A path in \( \overline{X} \) is a \( \alpha \)'s
weap \( I \xrightarrow{\gamma} \overline{X} \). A \( \alpha \) loop in \( \overline{X} \) is
a path s.t. \( \gamma(0) = \gamma(1) \), or equivalently
a weap \( s' \xrightarrow{\gamma} \overline{X} \).

A based loop in \( (\overline{X}, x_0) \) is
a weap \( (s', o) \xrightarrow{\gamma} (\overline{X}, x_0) \).

Def 10.8] If \( \gamma_1, \gamma_2 : I \to \overline{X} \) are two paths
in \( \overline{X} \) s.t. \( \gamma_1(0) = \gamma_2(0) \), we say there are composable.

form

a) \( \overline{\gamma}_1 : I \to \overline{X} \), the reverse path
\( \gamma_1(t) = \gamma_1(1-t) \)

b) \( \gamma_2 \times \gamma_1 \), the composite of \( \gamma_1, \gamma_2 \)
\[(r_2 * r_1)(t) = \begin{cases} r_1(2t) & \text{if } t \leq \frac{1}{2} \\ r_2(2t-1) & \text{if } t > \frac{1}{2}. \end{cases}\]

\(c) \quad e_x = \text{constant path of } x.\)

It is immediate that \(\tilde{\gamma} = \gamma,\) and not hard to see \(\tilde{\gamma}_1 \times \tilde{\gamma}_2 = \tilde{\gamma}_2 \times \tilde{\gamma}_1.\)

**Def 10.9** If \(\gamma : I \rightarrow X\) is a path, \\

\[\text{while } [\gamma] \text{ for the set of paths } \gamma \text{ s.t. } \delta_{\{0,1\}}(\gamma) = \gamma_{\{0,1\}}, \text{ and } \gamma \equiv \delta_{(v, e)}(v, e) \]

**Prop 10.10** The following hold

1. If \([\gamma] = [\gamma']\) and \([\delta] = [\delta']\) and \(\gamma, \delta\) are composable, then \([\gamma \times \delta] = [\gamma' \times \delta']\).

2. If \(\gamma, \delta, \epsilon\) are composable in this
\[
[(\delta \times \delta) \times 2] = [\gamma \times (\delta \times 2)]
\]

3. \[\gamma \times \gamma] = [e_{\gamma(1)}].

4. \[\gamma \times e_{\gamma(1)}] = [\gamma]
   \[e_{\gamma(1)} \times \gamma] = [\gamma]

Proof:

1. \]

2. In each case we give a handy
   rel \((s^0, i^1), 1\)
   are map \(I \to X\)
   to another
Def 10.1) Let \( [Y] \times [S] \) for the class \( [Y \times S] \). Let \( \Lambda(X) \) denote the set of all classes \( [Y] \) in \( X \). It is called the fundamental groupoid of \( X \).

Prop 10.12) If \( \phi : \overline{X} \to \overline{Y} \) is connected, and \( \gamma : I \to \overline{X} \) is a path, then define \( \phi_x \cdot [\gamma] \) by \( \phi \cdot \gamma [\gamma] \).

Then \( \phi_x \cdot [\gamma] \cdot \phi_x \cdot [\delta] = \phi_x \cdot [\gamma \times \delta] \).

Proof 10.4) ensures \( \phi_x \cdot [\gamma] \) is well-defined. The verification of the other properties is routine.
Def 10.13] Let \((\overline{X}, x_0)\) be a based space. Define \(\pi_1(\overline{X}, x_0)\) as
the set of equivalence classes \([\gamma]\) where \(\gamma(0) = \gamma(1) = x_0\).

Note all such paths are composable.

Prop 10.14] Under the composition law
\([\gamma] \times [\delta] = [\gamma \times \delta]\), the inverse
\([\gamma]^{-1} = [\overline{\gamma}]\) and with the identity
\([e_{x_0}] = e\), the set \(\pi_1(\overline{X}, x_0)\) is
a group, called the fundamental group
of \((\overline{X}, x_0)\).

Proof: This is Prop 10.10 in the
special case of paths all starting \&
ending at \( x_0 \).

**Prop 10.15** If \( f : (X, x_0) \to (Y, y_0) \) is a map, then we may define a group homomorphism

\[
f_\ast : \pi_1(X, x_0) \to \pi_1(Y, y_0)
\]

by

\[
f_\ast [g] = [f \circ g]
\]

**Proof:** This is 10.12 in the case that \( g \) starts \( f \) ends at \( x_0 \).

**Prop 10.16** \( \pi_1 \) is a **functor** from Top, the category of based topological spaces, to Grp, the category
of groups (or homomorphisms)

Proof: Objects: \((\bar{X}, x_0) \rightarrow \pi_1(\bar{X}, x_0)\)

Morfisms: \(\bar{X} \stackrel{f}{\rightarrow} \bar{Y} \rightarrow \pi_1(\bar{X}, x_0) \rightarrow \pi_1(\bar{Y}, y_0)\)

Functionality: i) identity - morfisms

\[ g_\bar{X}(f \cdot g) = [g_\bar{O}(f \circ g)] = (g \circ f)_\bar{X} \bar{Y} \]
Example 10.17) Trivial example

Let \((D, x_0)\) denote a discrete space with at least one point \((x_0)\).
Then \(\pi_1(D, x_0) = \{e\}\). (This applies in particular to \(\{x_0\}\))

In fact the argument works just as well anytime the path component of \(x_0\)
is a 1-point set.

Example 10.18)

We will prove later that \(\pi_1(S^1, o) \cong \mathbb{Z}\)

\[ \text{winding number} \]
Construction 10.19: Let \( X \) be a space and let \( \gamma : I \to X \) be a path in \( X \) with \( \gamma(0) = x_0 \), \( \gamma(1) = x \).

Let \( [\alpha] \in \pi_1(X, x_0) \), then we obtain an element of \( \pi_1(X, x) \) by

\[
[\alpha] \to [\gamma] \star [\alpha] \star [\tilde{\gamma}]
\]
two observations

\[ \phi_y ([\alpha] \times [\beta]) = \phi_y ([\gamma] \times [\alpha] \times [\beta] \times [\tilde{\gamma}]) = \phi_y ([\alpha]) \times \phi_y ([\beta]) = \phi_y ([\alpha]) \times \phi_y ([\beta]). \]

So \( \phi_y \) is a group homomorphism.

\[ \phi_{\tilde{g}} \circ \phi_y ([\alpha]) = \phi_{\tilde{g}} ([\gamma] \times \phi_y ([\alpha]) = \phi_{\tilde{g}} \circ \phi_y ([\alpha]) = [\alpha] \]

\[ \Rightarrow \phi_{\tilde{g}} \circ \phi_y = \text{id}. \]

So \( \phi_{\tilde{g}} \) is an isomorphism of groups.
Def 10.20 \text{ If } f : \overline{X} \rightarrow Y \text{ is a}
\text{ map, then we say } f \text{ is a}
\text{ homotopy equivalence if there exists}
\text{ a map } g : Y \rightarrow \overline{X} \text{ such that}
g \circ f \sim \text{id}_{\overline{X}}, \quad f \circ g \sim \text{id}_{Y}.
\text{g is a homotopy inverse.}

\text{Example: } S^1 \subseteq \mathbb{R}^2 - \{0\} \text{ is a homotopy equivalence.}

\text{To find } g, \text{ make points in } \mathbb{R}^2 - \{0\}
\text{ as } (r \cos \theta, r \sin \theta), \quad \theta \in [0, 2\pi), \quad r > 0
\text{ then define}
g : (r \cos \theta, r \sin \theta) \mapsto (\cos \theta, \sin \theta)
g \circ f = \text{id}_{\overline{X}}; \quad (f \circ g) (r \cos \theta, r \sin \theta) =
(\cos \theta, \sin \theta)

H (r \cos \theta, r \sin \theta, t) = (r^{(1-t)} \cos \theta, r^{(1-t)} \sin \theta)
H (x, y, \theta) = (x, y) = \text{id}_{\mathbb{R}^2 - \{0\}}
Any homeomorphism is a homotopy equivalent. \( f \circ f^{-1} = \text{id}_Y \quad f^{-1} \circ f = \text{id}_X \)

**Proposition 10.20** Let \( f, g : X \to Y \) be homotopic maps & let \( h(x_0) = y_0 \), \( h(x_1) = y_1 \). Then there is a path \( r : I \to Y \) such that

\[
\begin{array}{ccc}
\pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\
\phi \downarrow & & \phi \\
\pi_1(Y, y_1) & \xrightarrow{g_*} & \pi_1(Y, y_1)
\end{array}
\]

commutes.

**Proof**: Let \( H(x, t) \) be a homotopy from \( f \) to \( g \); let

\[
y(t) = H(x_0, t).
\]

Then for any \( [x] \in \pi_1(X, x_0) \)

\[
\begin{align*}
f_* [x] &= [x \circ f] \\
g_* [x] &= [x \circ g]
\end{align*}
\]
\( \phi_\alpha(f \times \alpha)](x) = [\gamma] \times [\alpha \circ f] \times [\gamma] \)

Want to show this is homotopic to \([\alpha \circ g] \).

Define three paths in \( \overline{\mathbb{R}} \times I \)

by \( \alpha_0(t) = (\gamma(t), 0) \)

\( \alpha_1(t) = (\gamma(t), 1) \)

\( \Gamma(t) = (x_0, t) \)

Then \( H \circ \alpha_0 = f \circ \gamma \)

\( H \circ \alpha_1 = g \circ \gamma \)

\( H \circ \Gamma = \gamma \)

Define \( F : I \times I \rightarrow \overline{\mathbb{R}} \times I \)
Let $\varepsilon_i$, $\Theta_i$ be the indicated paths in $I \times I$. Then $[\varepsilon_0 \times \Theta_i]$ is homotopic (rel endpunts) to $[\Theta_0 \times \varepsilon_i]$

\[ F^* (\varepsilon_0 \times \Theta_i) \simeq F^* (\Theta_0 \times \varepsilon_i) \]

\[ [\alpha_0 \times \Pi] \simeq [\Pi \times \alpha_i] \]

\[ h^* (\alpha_0 \times \Pi) \simeq h^* (\Pi \times \alpha_i) \]

\[ [f \alpha \times y] \simeq f \gamma \times g \circ \alpha \]
Prop 10.21: If \( f : X \to Y \) is a homotopy equivalence and \( f(x_0) = y_0 \), then
\[
f'_* : \pi_1(\overline{X}, x_0) \sim \pi_1(Y, y_0)
\]
is an isomorphism.

Proof: Let \( g \) be a homotopy inverse
\[
\pi_1(\overline{X}, x_0) \xrightarrow{f'_*} \pi_1(Y, y_0) \xrightarrow{g'_*} \pi_1(X, g(y_0))
\]
by 10.21 and the fact that \( g \circ f \sim \text{id}_\overline{X} \).
Prop 10.22 \hspace{1cm} \text{If } X \rightarrow Y \text{ is a hitpy equivalence and } Y \rightarrow Z \text{ is too, then } g \circ f \text{ is a hitpy equivalence.}

Proof \hspace{1cm} \text{let } h, i \text{ be hitpy inverses for } f, g \text{ respectively. Then } (g \circ f) \text{ has } (i \circ h) \text{ as a hitpy inverse. (Check).} \hspace{1cm} \blacksquare
Def 10.23 | Let \( A \hookrightarrow X \) be a subset. \( A \) is a (strong) deformation retract of \( X \) if there exists a map 
\[ f : \overline{A} \rightarrow A \] such that:
1. \( f \big|_A = \text{id}_A \)
2. \( i \cdot f = \text{id}_{\overline{X}} \) (oid \( A \)).

Prop 10.24 | If \( A \hookrightarrow X \) is a deformation retract, and \( a \in A \), then 
\[ i : (A, a) \rightarrow (\overline{X}, a) \] is a (based) homotopy equivalence.

Proof
\[
\begin{array}{c}
A \xrightarrow{i} \overline{X} \xrightarrow{f} A \\
\xrightarrow{\text{id}}
\end{array}
\]
Example 10.25] We say $A \subset \mathbb{R}^n$ is **starshaped** w.r.t. $\bar{a} \in A$ if, whenever $\bar{x} \in A$, the points

$$\lambda(\bar{x} - \bar{a}) + \bar{a} \in A \quad \forall \lambda \in [0, 1],$$

$\mathbb{R}^n$ is starshaped relative to any point.

$B_p(\bar{a}, \epsilon)$ are starshaped, $B_p(\bar{a}, \epsilon)$ also, relative to $\bar{a}$. 
Any convex body is starshaped.

If \( \overline{X} \) is starshaped w.r.t. \( \overline{x} \in \overline{X} \), then \( \{x\} \hookrightarrow \overline{X} \) is a deformation retract:

\[
\overline{X} \rightarrow \{x\} \rightarrow \overline{X},
\]

\[
F(y, t) = t(\overline{x} - \overline{x}) + \overline{x}.
\]

Therefore, in particular, \( \pi_1((x, \overline{x})) \) is
when $\overline{X}$ is a convex and $\overline{x} \in \overline{X}$ is any point. In particular $\pi_0(\mathbb{R}^n, \overline{x}) = e$. 

\[ \forall n. \]

**Definition 10.26** Let $\pi_0(\overline{X})$ be the set of path components of $\overline{X}$ (c.f. 6.16)

If a space $\overline{X}$ is such that

1. $\pi_0(\overline{X}) = e \times 1$
\[ \pi_1(\bar{X}, x_0) = \{e_{x_0}\} \]

then we say \( \bar{X} \) is simply connected.

(\( \bar{X} \) path connected if every loop is contractible to a point).

Prop 10.27 If \( \bar{X} \) is simply connected, then two paths in \( \bar{X} \) are homotopic (over endpoints) if they have the same endpoints.