I declare \( [0, 1] \)

**Def 10.1** If \( X, Y \) are spaces and \( A \subset X \) is a subspace (generally closed), and \( f, g : X \to Y \) are two \( C^0 \) functions such that \( f|_A = g|_A \) (i.e. \( f(x) = g(x) \) \( \forall x \in A \))
a **homotopy from \( f \) to \( g \) relative to \( A \) is a \( C^0 \) function \( H : X \times [0, 1] \to Y \) such that

\[
H(x, t) = \begin{cases} f(x) & \text{if } t = 0 \\ g(x) & \text{if } t = 1 \\ f(x) = g(x) & \text{if } x \in A. 
\end{cases}
\]

A homotopy relative to \( \varnothing \) is just called a homotopy.

If \( X, A \) are \( C^0 \)

\( C(X \times [0, 1], Y) \cong C([0, 1], C(A, Y)) \)

\( C(A \times [0, 1], Y) \cong C([0, 1], C(A, Y)) \)

then a homotopy \( H : X \times [0, 1] \to Y \) rel. \( A \)
is equivalent to a path \( \gamma : I \to C(A, Y) \)
such that the composite \( \tilde{\gamma} : I \to C(X, Y) \to C(X) \)
The maps \( f, g \) are **homotopic**.

we write \( f \simeq g \) (rel \( A \))

\[
\text{Prop 10.2: } \simeq \text{ is an equivalence relation on maps } X \rightarrow Y, \text{ i.e.} \\
i \ f \simeq f \\
(ii) \ f \simeq g \Rightarrow g \simeq f \\
(iii) \ f \simeq g, \ g \simeq h \Rightarrow f \simeq h. \text{ (rel } A) \]

**Proof**: i) is easy 

   ii) 
   given \( H : X \times I \rightarrow Y \) giving \( f \simeq g \), define \( \hat{H} : X \times I \rightarrow Y \) by

\[
- \hat{H} (x, t) = H(x, 1-t)
\]
iii) given \( h_1 \) from \( f \) to \( g \),
\( h_2 \) from \( g \) to \( h \)

define \( h_1 \times h_2 : X \times I \rightarrow Y \) by

\[
(h_1 \times h_2)(x, t) = \begin{cases} 
    f h_1 (x, 2t) & ; t \leq \frac{1}{2} \\
    f h_2 (x, 2t - 1) & ; t > \frac{1}{2}
\end{cases}
\]

\( \text{Def 10.3) The set of maps } g \text{ st. } \)
\( g \circ f \) is called the homotopy class of \( f \). We write \( \text{hlo}(X, Y) \) for the set of homotopy classes of maps.
10.21 Let $X, Y, Z$ be top. sp. $A \subseteq Z$ $f_1, f_2 : X \to Y$ s.t. $f_1|_A = f_2|_A : B \to Y$ s.t. $f_1|_A \subseteq B$ & $g_1|_B = g_2|_B$ then if $f_1 \simeq f_2$ (rel $A$) & $g_1 \simeq g_2$ (rel $B$) then $g_1 \circ f_1 \simeq g_2 \circ f_2$ (rel $A$).

Proof: Let $H_1$ be a homotopy from $f_1$ to $f_2$ & $H_2$ a homotopy from $g_1$ to $g_2$ then $H : \Delta \times I \to Z$

\[ H(x, t) = H_2(0, (x, t), t) \]

This yields the required hompy. \[ \Box \]

**Based spaces**

**Def 10.5** A **based or pointed space** is a space $X$ equipped with a distinguished element $* \in \overline{X}$, or equivalently a distinguished map $\{ * \} \times I \to \overline{X}$ to one-point space
A map of based spaces \((\overline{X}, x_0) \rightarrow (Y, y_0)\)
is a \(C^0\) map \(f: \overline{X} \rightarrow Y\) s.t. \(f(x_0) = y_0\)
equivalently \[
\begin{array}{c}
x_0 \\
\overline{X} \\
\end{array} \xymatrix{
\ar[r]^f & \ar[d] \ar[r] & Y \\
\cdot} \quad \text{for} \quad \begin{array}{c}
x_0 \\
\overline{X} \\
\cdot
\end{array}
\]

When we need it to be based, we will base \(X\) at \(x_0\). \(S' = I / \{0, 1\}\) will be also based here.

\[\text{Def 10.61} \quad \text{If} \quad (X, x_0), (Y, y_0) \text{ are based spaces, a based homotopy of maps} \ f_1, f_2 \text{ is a map}
\]
\[H : \overline{X} \times I \rightarrow Y
\]
\[\text{based at} \quad (x_0 \times 0)
\]
s.t. \(H\) is a homotopy from \(f_1\) to \(f_2\) rel \(x_0\)
\[\text{i.e.} \quad H(x_0, t) = y_0 \quad \forall t.
\]
The set of based homotopy classes is written \([\Delta, x_0], (y, y_0)\) or \([\Delta, y]\) when
\(x_0, y_0\) are clear.
Def 10.7 | A path in $\mathbb{X}$ is a cts map $I \xrightarrow{\gamma} \mathbb{X}$. A loop in $\mathbb{X}$ is a path s.t. $\gamma(0) = \gamma(1)$, or equivalently a map $s' \xrightarrow{\gamma} \mathbb{X}$. A based loop in $(\mathbb{X}, x_0)$ is a cts map $(s', 0) \xrightarrow{\gamma} (\mathbb{X}, x_0)$.

Def 10.8 | a) If $\gamma : I \to \mathbb{X}$ is a path, we define the reverse path of $\gamma$, $\tilde{\gamma} : I \to \mathbb{X}$ by $\tilde{\gamma}(t) = \gamma(1-t)$.

b) If $\gamma_1, \gamma_2$ are two paths in $\mathbb{X}$ s.t. $\gamma_1(0) = \gamma_2(0)$, we say $\gamma_1, \gamma_2$ are composable and define the composite path $\gamma_1 \times \gamma_2 : I \to \mathbb{X}$ by
\((x_1 \star x_2)(t) = \begin{cases} x_1(2t) & \text{if } t \leq \frac{1}{2} \\ x_2(2t-1) & \text{if } t \leq \frac{1}{2} \end{cases}\)

c) \(e_x = \text{constant path of } x\).

It is immediate that \(\tilde{\gamma} = \gamma\), and we need to see \(\tilde{\gamma}_1 \times \tilde{\gamma}_2 = \tilde{\gamma}_2 \times \tilde{\gamma}_1\).

**Def 10.9** If \(\gamma : I \rightarrow X\) is a path, define \([\gamma]\) for the set of paths \(\delta\) s.t.

\[\delta\{0, 1\} = \gamma\{0, 1\}\] and \(\gamma = \delta\) (verbal)

**Prop 10.10** The following hold

1. If \([\gamma] = [\gamma']\) and \([\delta] = [\delta']\) and \(\gamma, \delta\) are composable, then
   \([\gamma \times \delta] = [\gamma' \times \delta']\).

2. If \(\gamma, \delta, \varepsilon\) are composable in this
$$\text{order} \quad \{ (\gamma \times \delta) \times \epsilon \} = \{ \gamma \times (\delta \times \epsilon) \}$$

3. \( \{ \gamma \times \bar{\gamma} \} = \{ e_{\gamma(1)} \} \).

4. \( \{ \gamma \times e_{\gamma(1)} \} = \{ \gamma \} \)

\( \text{Proof:} \)

There is a homotopy \( H_\gamma : I \times I \rightarrow \bar{X} \) of \( \gamma \) to \( \gamma' \), \( H_\delta : I \times I \rightarrow \bar{X} \) of \( \delta \) to \( \delta' \) (rel \( 0, 1 \)). Define

\[
G : \bar{I} \times \bar{I} \rightarrow \bar{X}
\]

by

\[
G(s, t) = \begin{cases} 
H_\delta(2s, t) & \text{if } s \leq \frac{1}{2} \\
H_\gamma(2s-1, t) & \text{if } s > \frac{1}{2}
\end{cases}
\]

2. \( L(\gamma, t) = \begin{cases} \gamma \left( \frac{2s}{1+t} \right) & 0 \leq s \leq \frac{t+1}{t_c} \\
\delta \left( \frac{4s-t-1}{t_c} \right) & \frac{t+1}{t_c} \leq s \leq \frac{t+2}{t_c} \\
\epsilon \left( \frac{(4s-t-2)(2-t_1)}{t_c} \right) & \frac{t+2}{t_c} \leq s \leq 1
\end{cases} \)
Define \( H(s,t) = \begin{cases} \gamma(2s); & 0 \leq s \leq \frac{1-t}{2} \\ \gamma(1-t); & \frac{1-t}{2} < s \leq \frac{1+t}{2} \\ \gamma(2-2s); & \frac{1+t}{2} < s \leq 1 \end{cases} \).

4. \( H'(s,t) = \begin{cases} \gamma\left(\frac{2s}{1+t}\right); & 0 \leq s \leq \frac{1-t}{2} \\ \gamma(t); & \frac{1+t}{2} < s \leq 1 \end{cases} \).

The other case is similar. \( \square \)

Munkres observes that any
Def 10.1.1. Let \([Y] \times [S]\) be the class \([y \times S]\). Let \(\Lambda(\tilde{X})\) denote the set of all classes \([y]\) in \(\tilde{X}\).
It is called the fundamental groupoid of \(\tilde{X}\).

Prop 10.12. If \(\phi : \tilde{X} \to \tilde{Y}\) is c'k, and \(\gamma : I \to \tilde{X}\) is a path, then define \(\phi \times [\gamma] \) by \([\phi \circ \gamma]\).
Then \(\phi \times [\gamma] \times \phi \times [S] = \phi \times [y \times S]\)
\(\phi \times [\gamma] = \phi \times [\gamma]\)
Proof: 10.4 ensures \(\phi \times [\gamma]\) is well-defined. The verification of the other properties is not difficult.
Def 10.13] Let \((\overline{X}, x_0)\) be a based space. Define \(\pi_1(\overline{X}, x_0)\) as the set of equivalence classes \([\gamma]\) where \(\gamma(0) = \gamma(1) = x_0\).

Note all such paths are composable.

Prop 10.14] Under the composition law

\([\gamma] \times [\delta] = [\gamma \circ \delta]\), the inverse

\([\gamma]^{-1} = [\overline{\gamma}]\) and with the identity

\([e_{x_0}] = e\), the set \(\pi_1(\overline{X}, x_0)\) is a group, called the fundamental group of \((\overline{X}, x_0)\).

Proof: This is Prop 10.10 in the special case of paths all starting \&
ending at \( x_0 \).

\[ \text{Def/Prop 10.15} \] If \( f : (X, x_0) \rightarrow (Y, y_0) \) is a \( f \)'s map, then we may define a group homomorphism

\[ f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0) \]

by \( f_*[\gamma] = [f \circ \gamma] \).

Proof: This is 10.12 in the case that \( \gamma \) starts \( f \) and \( \gamma \) ends at \( x_0 \).

\[ \text{Prop 10.16} \] \( \pi_1 \) is a functor from \( \text{Top} \), the category of based topological spaces, to \( \text{Grp} \) the category
Proof: Objects: $(\bar{X}, x_0) \rightarrow \bar{\pi}, (\bar{X}, x_0)$

Morphisms: $\bar{X} \xrightarrow{f} \bar{Y} \rightarrow \bar{\pi}, (\bar{X}, x_0) \xrightarrow{f_x} \bar{\pi}, (Y, Y_0)$

Functionality:

i) identity - domains

$g_x(f \times \{y\}) = [g_0(f \circ y)] =

f(g \circ f) \circ g = (g \circ f)(\bar{x}, \bar{y})$
Example 10.17) Trivial example
Let \((D, x_0)\) denote a discrete space with at least one point \((x_0)\).
Then \(\pi_1(D, x_0) = 1\).
(This applies in particular to \(\{x_0\}\))
In fact the argument works just as well anytime the path component of \(x_0\)
is a 1-point set.

Example 10.18)
We will prove later that
\(\pi_1(\gamma, 0) \simeq \mathbb{Z}\)
\(\text{winding number}\)
Construction 10.19: Let \( X \) be a space and let \( \gamma: I \to X \) be a path in \( X \) with \( \gamma(0) = x_0 \), \( \gamma(1) = x_1 \).

Let \([\alpha] \in \pi_1(X, x_0)\), then we obtain an element of \( \pi_1(X, x_1) \) by
\[
[\alpha] \mapsto [\gamma] \star [\alpha] \star [\widetilde{\gamma}]
\]
two observations

\[ \phi_x([\alpha] \times [\beta]) = [\gamma] \times [\alpha] \times [\beta] \times [\check{\gamma}] \]
\[ = [\gamma] \times [\alpha] \times [\check{\gamma}] \times [\check{\beta}] \times [\check{\gamma}] \]
\[ = \phi_x([\alpha]) \times \phi_x([\beta]). \]

so \( \phi_x \) is a group homomorphism.

\[ \phi_y \circ \phi_x([\alpha]) = [\gamma] \times [\gamma] \times [\alpha] \times [\check{\gamma}] \times [\check{\gamma}] \times [\check{\gamma}] \]
\[ = [\alpha] \]
\[ = \phi_y \circ \phi_x = id. \]

so \( \phi_y \) is an isomorphism of groups.
Def 10.20: If \( f : X \to Y \) is a map, then we say \( f \) is a **homotopy equivalence** if there exists a map \( g : Y \to X \) such that

\[ g \circ f \simeq \text{id}_X \quad \text{and} \quad f \circ g \simeq \text{id}_Y. \]

\( g \) is a **homotopy inverse**.

Example: \( S^1 \subseteq \mathbb{R}^2 - \{0\} \) is a homotopy equivalence.

To find \( g \), make points in \( \mathbb{R}^2 - \{0\} \) as \( (r \cos \theta, r \sin \theta) \quad \theta \in [0, 2\pi), \ r > 0 \)

then define

\[ g : (r \cos \theta, r \sin \theta) \mapsto (\cos \theta, \sin \theta) \]

\[ g \circ f = \text{id}_X \quad ; \quad (f \circ g) (r \cos \theta, r \sin \theta) = (\cos \theta, \sin \theta) \]

\[ H \left( r \cos \theta, r \sin \theta, t \right) = \left( r^{(1-t)} \cos \theta, r^{(1-t)} \sin \theta \right) \]

\[ H \left( x, y, \theta \right) = \left( x, y, \theta \right) \quad ; \quad \text{id}_{\mathbb{R}^2 - \{0\}} \]
Any homeomorphism is a homotopy equiv
\[ f \circ f^{-1} = 1 \quad 1 \circ f = f \circ g \]

Prop 10.20: Let \( f, g : X \to Y \) be homotopic maps and let \( h(x_0) = y_0 \), \( h(x_1) = y_1 \). Then there is a path \( r : I \to Y \) such that

\[ \pi_1(\bar{X}, x_0) \xrightarrow{f} \pi_1(Y, y_0) \]
\[ g \raise 0.5ex \big| \phi \]
\[ \gamma \]
\[ \pi_1(Y, y_1) \]

commutes

Proof: Let \( H(x, t) \) be a homotopy from \( f \) to \( g \); let

\[ \gamma(t) = H(x_0, t) \]

then for any \( [x] \in \pi_1(\bar{X}, x_0) \)
\[ \begin{align*}
    f_\gamma [x] &= [x \circ f] \\
    g_\gamma [x] &= [x \circ g]
\end{align*} \]
\( \phi_g(f \times \alpha) = [\gamma] \circ [\alpha \circ f] \times [\gamma] \)

want to show this is homotopic to 
\([\alpha \circ g]\).

Define three paths in \( \overline{X} \times I \) 

by 
\[
\begin{align*}
\alpha_0(t) &= (\gamma(t), 0) \\
\alpha_1(t) &= (\gamma(t), 1) \\
\Gamma(t) &= (x_0, t)
\end{align*}
\]

Then \( H \circ \alpha_0 = f \circ \gamma \)
\( H \circ \alpha_1 = g \circ \gamma \)
\( H \circ \Gamma = \gamma \)

Define \( F : I \times I \rightarrow \overline{X} \times I \)
Let $\beta_i$, $\theta_i$ be the indicated paths in $I \times I$. Then $[\varepsilon_0 \times \theta_i]$ is homotopic (rel endpoints) to $[\theta_0 \times \varepsilon_i]$

$\Rightarrow F_\times [\varepsilon_0 \times \theta_i] \simeq F_\times [\theta_0 \times \varepsilon_i]$

$\Rightarrow [\alpha_0 \times \Pi] \simeq [\Pi \times \alpha_i]$

$\Rightarrow H_\times [\alpha_0 \times \Pi] \simeq H_\times [\Pi \times \alpha_i]$

$\Rightarrow [\alpha \times \gamma] \simeq \gamma \times g \circ \alpha$
Prop 10.21 \[ \text{If } f : X \to Y \text{ is a homotopy equivalence and } f(x_0) = y_0, \]
\[ f_* : \pi_1 (X, x_0) \cong \pi_1 (Y, y_0) \]
\[ \text{is an isomorphism.} \]

Proof: Let \( g \) be a homotopy inverse \( \pi_1 (X, x_0) \xrightarrow{f_*} \pi_1 (Y, y_0) \xrightarrow{g_*} \pi_1 (X, g(y_0)) \)
\[ \xrightarrow{\text{id}} \]
\[ \pi_1 (X, x_0) \]
by 10.21 and the fact that \( g \circ f \cong \text{id}_X \).
Prop 16.22 | If \( \bar{x} \rightarrow y \) is a

hypo equivalence and \( y \rightarrow \bar{z} \) is too,

then \( g \circ f \) is a hypo equivalence.

Proof | Let \( h \), \( i \) be hypo inverses

for \( f \), \( g \) respectively. Then

\( (g \circ f) \) has \((i \circ h)\) as a

hypo inverse. (Check). \( \square \)
**Def. 10.23** Let $A \xrightarrow{i} \bar{X}$ be a subset. $A$ is a (strong) deformation retract of $\bar{X}$ if there exists a map $f : \bar{X} \to A$ s.t.

1. $f|_A = \text{id}_A$

2. $i \cdot f = \text{id}_{\bar{X}}$ (or $A$).

**Prop 10.24** If $A \xrightarrow{i} \bar{X}$ is a deformation retract, and $a \in A$, then $i : (A, a) \to (\bar{X}, a)$ is a (based) homotopy equivalence.

**Proof**

\[
A \xrightarrow{i} \bar{X} \xrightarrow{f} A
\]

\[
\text{id}
\]
Example 10.25] We say $A \subset \mathbb{R}^n$ is starshaped w.r.t. $\bar{a} \in A$ if, whenever $\bar{x} \in A$, the points 

$$\lambda (\bar{x} - \bar{a}) + \bar{a} \in A \quad \forall \lambda \in [0,1].$$

$\mathbb{R}^n$ is starshaped relative to any point. $B_p(\bar{a}, \varepsilon)$ are starshaped, $B_p(\bar{a}, \varepsilon)$ also, relative to $\bar{a}$. 
A starshaped region.

Any convex body is starshaped.

If \( \bar{x} \) is starshaped w.r.t. \( \bar{x} \in \mathcal{X} \), then \( \{x\} \hookrightarrow \bar{x} \) is a deformation retract:

\[
\bar{x} \mathrel{\hookrightarrow} \{x\} \rightarrow \bar{x} \quad \text{id}
\]

\[
T(y, t) = t(\bar{y} - \bar{x}) + \bar{x}
\]

Therefore, in particular, \( \pi_1(\bar{x}, \bar{x}) \approx \mathbb{E} \).
when \( \overline{X} \) is no convex and \( \pi \in \overline{X} \) is any point. In particular \( \pi_0((\mathbb{R}^n, \overline{X})) = \mathbb{E} \) for \( \forall n \).

**Definition 10.26** Unite \( \pi_0(\overline{X}) \) for the set of path components of \( \overline{X} \) (c.f. (6.16))

If a space \( \overline{X} \) is such that

\[ \pi_0(\overline{X}) = \emptyset \]

2. $\pi_1(\overline{X}, x_0) = \{[e_{x_0}]\}$

then we say $\overline{X}$ is simply connected.

(\overline{X}$ is path connected if every loop is contractible to a point).

Prop 10.27 If $\overline{X}$ is simply connected, then two paths in $\overline{X}$ are homotopic (over endpoints) if they have the same endpoints.
Part II of Ch 10

Homotopy and Covering Spaces.

Def 10.28 | A map \( f : X' \to X \) is a covering space if it is surjective.
and if \( \forall x \in \tilde{X}, \exists U \ni x \) open s.t. \( f^{-1}(U) = \bigsqcup_{i \in I} U_i \) where the \( U_i \) are disjoint \& \( f|_{U_i} : U_i \to U \) is a homeomorphism \( \forall i \).

\( \tilde{X} \) is said to cover \( X \).

**Example 10.29**

\[ a) \quad \mathbb{R} \to \mathbb{S}^1 \]
\[ x \mapsto (\cos x, \sin x) \]

is a covering space.

\[ (-\infty, -\infty) \quad (-\infty, -\infty) \]

\[ \bigcirc \]

\[ b) \quad \mathbb{S}^1 \xrightarrow{x^n} \mathbb{S}^1 \] by \( m \in \mathbb{Z} \)
\[ (\cos \theta, \sin \theta) \mapsto (\cos m\theta, \sin m\theta) \]
is a covering space.

c) The disjoint union of two covering spaces is a covering space.

d) $\mathbb{R}^2 \rightarrow \mathbb{R}^2 / \mathbb{Z}^2$
Notah 10.30 \quad \text{Given a map}

\[ \begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow p & & \downarrow p \\
Y & \xrightarrow{f} & X
\end{array} \]

a lift is a map \( f' \) making the diagram commute.

Prep 10.31 \quad \text{Given a covering space}

\[ p : X' \rightarrow X, \text{ a deck } f : Y \rightarrow X, \]

a lift \( f' : Y \rightarrow X' \) and

a homotopy \( H : f \sim g \)

\[ \begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow f' & & \downarrow p \\
Y \times I & \xrightarrow{H} & X
\end{array} \]
there exists a unique lift $\tilde{f'}$ as indicated.

**Proof:** Fix a cover $\{U_x\}_{x \in \bar{X}}$ such that $\tilde{p}^{-1}(U_x)$ consists of a disjoint union of open sets in $\mathbb{R}$, each one homeomorphic via $\tilde{p}$ to $U_x$. Let $y \in Y$ and consider all the points $f(y, t) \in \bar{X}$. We may find product neighbourhoods $(y, t) \in N_t \times (a_t, b_t)$ such that $f(N_t \times (a_t, b_t)) \subset U_x$ for some $U_x$. Since $I$ is compact, we can choose finitely
wef of these $N \times (a_t, b_t)$ that cover $\{y\} \times I$. Intersect the 
(finitely many) $N_t$'s in this collection 
to give $N \ni y$ open.

In particular, we may write 
down finitely many sets

$$N \times [t_i, t_{i+1}] \quad \text{s.t.}$$

$$U_i \cap (N \times [t_i, t_{i+1}]) \text{ lies in } U_i,$$

which is in the cover $\{U_i\}$ of $\hat{x}$

$$0 = t_0 < \ldots < t_m = 1$$

\[ \text{Diagram} \]
Assume $H'$ has been constructed on $N \times [0, t_i]$ and try to extend to $N \times [0, t_{i+1}]$. (induct)
(Note $H'$ has been constructed on $N \times [0, t_i]$, it's $f'$).

Consider $\forall l (N \times [t_i, t_{i+1}]) \subseteq U_i$

and $H'(l(y, t_i))$ which in $p^{-1}(U_i) = \bigsqcup_{j \in S} U_{i,j}$

$\Rightarrow H'(l(y, t_i))$ lies in some set $U_{i'}$, that is homeomorphic (by $p$) to $U_i$. 
replace $N$ by

$$N \cap \left( \left( \left( t \mid f_t \right) \right)^{-1} \left( U_i \right) \right)_{N \times \{t; i\}}$$

a maybe smaller, but still open, nbhd of $y$. Then we define

$$f_t \mid_{N \times \{t; i\}}$$

to be the

nbhd.
$N \times [t_i, t_{i+1}] \xrightarrow{H} U_i \xrightarrow{g_i^{-1}} U'_i$

Thus agrees with what was previously constructed on $N \times [t_i, t_{i+1}]$.

So it's possible to lift $H$ in a small nbhd of any point $y \in U_i$. We show the lift is uniquely determined at any given $y \in U_i$.

Suppose $H'$ and $H''$ are two lifts of $g \times S \xrightarrow{h} X \xrightarrow{\pi}$ s.t. $H'(0) = H''(0) \Rightarrow \pi$

As before, we can find

$0 = t_0 < \ldots < t_{m+1} = 1$

s.t. $H([t_i, t_{i+1}])$ is contained
in the space $U_i$. Assume inductively
\[ h'_\mid_{[0, t_i]} = h'' \mid_{[0, t_i]} \]

since $[t_i, t_{i+1})$ is connected, $h'(t_i, t_{i+1})$ must lie in a
single component of $p^{-1}(U_i)$

Similarly for $h''([t_i, t_{i+1})$.

\[ h \to U_i \]
\[ h' \rightsquigarrow \int \]
\[ [t_i, t_{i+1}) \to U \]

since $h'(t_i) = h''(t_i)$

it must be the case $U_i$!
and since \( P \) is a homomorphism, \( h \), \( h' \) must agree on \( \{ t_i, t_{i+1} \} \). This proves uniqueness of \( H' \) at a single \( y \).

For any \( y \in Y \), \( \exists N \in \mathbb{N} \) s.t. \( H' \) exists and is unique on \( N \times I \).

By uniqueness, we have defined a function \( H' \) on any pair \( (y, t) \in Y \times I \). Every point has a whole curve \( c' \) of \( H' \) is \( c'^{th} \) to \( H' \) is \( c'^{th} \).
Example 10.3.2

Let \( (X, x_0) \) be a based space & let \( \pi : (\overline{X}, x_0') \rightarrow (\overline{X}, x_0) \) be a covering map.

For every path \( \gamma : (I, 0) \rightarrow (\overline{X}, x_0) \)

there exists a unique lift \( \gamma' : (I, 0) \rightarrow (\overline{X}, x_0') \)

and if \( \gamma, \gamma' \) are two paths homeomorphic (fix end points) then their lifts are homeomorphic, fixing endpoints.
The first part is easy to prove.

\[ \text{2nd part: let } [\gamma] = \{\gamma_t\} \]

\[ H : I \times I \to X \]

\[ H(t, d) = \gamma(t) \]

\[ H(0, 0) = \gamma_0(t) \]

\[ \text{lift } \gamma(t), \text{ for } \gamma : I \to X \]

\[ \text{by part 1. These are unique.} \]
uniqueness implies 

\[ t' (t, 0) = \gamma' (0) \]

\[ t'(1, 1) = \gamma_1 (0) \]

since \[ t(1, 3) = \gamma(1) = \gamma_1 (1) \]

when we lift this to \( \overline{X} \) we get a constant map (by uniqueness)
Hence \( t' \) gives a path \( \gamma' \) with \( \gamma'(t) \approx \gamma(t) \).

Example 10.33

\[
\pi_1\left(S^1, 0\right) \cong \mathbb{Z}
\]

Use the covering map

\[
\begin{array}{cc}
\mathbb{R} & \downarrow \gamma \\
& \quad \downarrow \\
& \begin{array}{c}
\text{cover} \end{array}
\end{array}
\]
This is a covering space. Let \((1, 0)\) denote the basepoint of \(S^1\), let \(0\) denote the basepoint of \(\mathbb{R}\). Define paths \([u]\) as follows:

\[
\begin{align*}
I & \xrightarrow{\mathbf{x}_u} \mathbb{R} \xrightarrow{P} S^1 \\
\mathbf{t} & \mapsto \mathbf{n}_t \xrightarrow{P} \cos 2\pi u, \\
& \quad \sin 2\pi u
\end{align*}
\]
if \([N] = [m]\), then by \(10.30\) \([xn] = [xm]\) if they must end at the same point \(\Rightarrow n = m\).

This gives an injective map

\[ \mathcal{U} \longrightarrow \tilde{\pi}_1(S^1, (r, o)) \]

As for surjectivity, \(\tilde{\pi}_1([R, o]) = A\)
so any two paths in \([R, o]\) with the same endpoints are homotopic.

Let \([p] \in \tilde{\pi}_1(S^1, (1, o)) \)
then \( f \) lifts uniquely to a path \( f' \) in \( \pi_1 \), starting at \( 0 \) and ending at some integer

\[ \Rightarrow f' \cong x \cdot n : [0,1] \to \mathbb{R} \]

\[ \Rightarrow \pi_1(f') = [y] \]

\[ \cong [n] . \]

Strictly speaking, we’ve established a bijective map of sets

\[ \gamma \rightarrow (\pi_1, [S'], (1,0)) \]

We should verify that \([v, \gamma']\)
\[ \pi_1 \left( \tilde{X}, \tilde{x}_0 \right) \xrightarrow{P^*} \pi_1 \left( \tilde{X'}, \tilde{x}'_0 \right) \]

Suppose \((\tilde{X}, \tilde{x}_0) \xrightarrow{P} (\tilde{X'}, \tilde{x}'_0)\) is a covering map, \(\tilde{X}'\) path connected. Then \(\pi_1 \left( \tilde{X'}, \tilde{x}'_0 \right) \xrightarrow{P^*} \pi_1 \left( \tilde{X}, \tilde{x}_0 \right)\) is an injective map.

If there is a bijection

\[ \pi_1 \left( \tilde{X}, \tilde{x}_0 \right) \xrightarrow{\text{bijection}} \pi_1 \left( \tilde{X}, \tilde{x}_0 \right) \]

**Proof:** We just proved this in the circle case. To reiterate...
\( \tilde{X} \) \hspace{1cm} \text{other paths in } \tilde{p}'(x_0) \\
\hspace{1cm} \nu \hspace{1cm} \tilde{p}

\( \tilde{X} \) \hspace{1cm} \text{c} \hspace{1cm} \tilde{I} \\
\xrightarrow{p'} \pi_1(\tilde{X}', x_0) \longrightarrow \pi_1(\tilde{X}, x_0) \\
\text{is given by composition.}

A path \( \gamma : [0, 1] \longrightarrow (\tilde{X}, x_0) \) has (unique) lift \( \tilde{e}(\gamma) : [0, 1] \longrightarrow (\tilde{X}, x_0) \). This lifting of paths \( \gamma \circ \gamma_1 \) has a (unique) lift to a homotopy...
\[ l(\gamma) = l(\gamma') \implies \text{there exists a "lifting" function} \]

\[ l : \pi_1(\bar{X}, x_0) \rightarrow \prod(\bar{X}') \]

\( \Updownarrow \) fundamental groupoid.

(Actually all classes in fundamental groupoid at \( x_0' \))

Notably if \( \alpha \in \pi_1(\bar{X}', x_0') \) then

\[ l(p_\gamma'(\alpha)) = \alpha. \]

\( \Rightarrow p_\gamma' \) is injective. Hooray!

Given a class \( \alpha \in \pi_1(\bar{X}, x_0) \), it has a unique lift to a class \( \alpha' \in \prod(\bar{X}') \) where \( \alpha \) starts at \( x_0' \) and ends at some \( y \in p'(x_0) \). The endpoint \( y' \) defines
a map

\[ m : \pi_1(\overline{X}, x_0) \rightarrow \tilde{p}'(x_0) \].

Given any \( y \in \tilde{p}'(x_0) \), there is a path from \( x_0 \) to \( y \) (\( \tilde{X}, x_0 \) being \( \eta \) for the class of this path then \( p(x(\eta)) \in \pi_1(\overline{X}, x_0) \) has a lift \( \eta = x(\nu(\eta)) \) \( m(p(\nu(\eta))) = y \) \), so \( m \) is surjective.

Suppose finally that \( x, \beta \) both satisfy \( m(x) = m(\beta) = y \) then \( x \ast \beta \) lifts, by uniqueness
of lifts, to a path in \( \tilde{\pi}_1(\tilde{X}, x_0) \)

\[ = \alpha \ast \beta^{-1} \in \text{Im } \rho \]

\[ = \alpha = \text{Im } \rho \ast \beta \]

\[ \text{Cor: If } (\tilde{X}', x_0) \xrightarrow{\bar{f}} (\tilde{X}, x_0) \]

is a covering map if \( \tilde{X}' \) is path connected (\( = \) \( \tilde{X} \) is path connected)

then \( \left| \text{Im } \rho \right| \xrightarrow{\text{c.w.}} \left| \tilde{\pi}_1(\tilde{X}, x_0) \right| \)

does not depend on \( x_0 \).
Cor. 10.36

If \((\widetilde{X}, x_0)\) is a based, path-connected space or if \((\widetilde{X}, x_0)\) is a simply-connected space and \(p: (\widetilde{X}, x_0) \to (\widetilde{E}, x_0)\) is a covering map, then there is a bijection

\[ p^{-1}(x_0) \cong \pi_1(\widetilde{X}, x_0) \]
Example 10.37

There is no retraction of the closed dish \( D = \{ (x, y) \mid x^2 + y^2 \leq 1 \} \) onto its boundary circle \( S^1 = \{ (x, y) \mid x^2 + y^2 = 1 \} \).

If there was then if \( (S^1, p) \xrightarrow{\text{rel } s} (D, p) \xrightarrow{\text{ref } s} (S^1, p) \)

a homotopy of \( \pi_1 \), gives

\[ \mathbb{Z} \rightarrow X \rightarrow \mathbb{Z} \]

\[ \text{rel } \alpha \]
Prop 10.38. Let \((\overline{X}, x_0), (Y, y_0)\) be based spaces; there is a natural isomorphism
\[
\pi_1(\overline{X} \times Y, (x_0, y_0)) \to \pi_1(\overline{X}, x_0) \times \pi_1(Y, y_0)
\]

**Proof:**
The map is constructed as follows: there are projection maps
\[
\begin{array}{ccc}
\overline{X} \times Y & \xrightarrow{pr_1} & \overline{X} \\
& \searrow & \downarrow \leftarrow \nwarrow \quad pr_2 \\
& & Y
\end{array}
\]
(taking \((x, y_0)\) to \(x_0 \& y_0\), respectively). Then
\[
(pr_1)_* : \pi_1(\overline{X} \times Y, (x_0, y_0)) \to \pi_1(\overline{X}, x_0)
\]
This is given by sending a path \(\gamma\) to \(pr_1 \circ \gamma : I \to \overline{X} \times Y \to \overline{X}\).

Therefore there is a group homomorphism
\[
\psi : \pi_1(\overline{X} \times Y, (x_0, y_0)) \to \pi_1(\overline{X}, x_0) \times \pi_1(Y, y_0)
\]
given by \(\psi(\gamma) = (pr_1 \circ \gamma, pr_2 \circ \gamma)\).
Moreover, given two maps

\[ s : (I, 0) \rightarrow (\overline{A}, x_0) \quad \epsilon : (I, 0) \rightarrow (Y, y_0) \]

define \( q(\delta, \varepsilon) : (I, 0) \rightarrow (\overline{A} \times Y, (x_0, y_0)) \) by

\[ I \xrightarrow{\delta} \overline{A} \xrightarrow{\text{pr}_2} Y \]

\[ \xrightarrow{\epsilon} \overline{A} \times Y \]

\[ \xrightarrow{\text{pr}_1} Y \]

(universal property of the product.)

It's immediate that \( \text{pr}_1 \circ q(\delta, \varepsilon) = s \)

\[ \text{pr}_2 \circ q(\delta, \varepsilon) = \varepsilon \]

Therefore \( \psi \) is surjective.

It remains to show that if \( \psi(\gamma) = \text{Id} \gamma \) then \( \gamma = \text{Id} \gamma \).
\[ \pi'[\gamma] = \gamma[\gamma'] \text{ is equivalent to} \]
\[ \text{pr}_1 \times (1_\gamma) = \text{pr}_1 \times (1_{\gamma'}) \quad \text{and} \quad \text{pr}_2 \times (1_\gamma) = \text{pr}_2 \times (1_{\gamma'}) \]

This implies

Hence we can find homotopies

\[ \overline{X} \times \overline{Y} \longrightarrow \gamma \]

\[ \begin{array}{ccc}
I \times X & \xrightarrow{H_2} & X \\
\downarrow H_1 & & \downarrow \\
I \times I & \xrightarrow{H_1} & X \\
\end{array} \]

\[ H_1 \left(t, 0 \right) = \text{pr}_1 X \quad H_1 \left(t, 1 \right) = \text{pr}_1 X' \]

And similarly for \( H_2 \) - check endpoints.

\( \Rightarrow \) find a lift \( I \times I \longrightarrow \overline{X} \times \overline{Y} \) giving a homotopy from \( \gamma \) to \( \gamma' \). \( \square \)

Cor 10.3.9 \[ \pi_1 \left( S^1 \times S^1 \times \ldots \times S^1, * \right) \cong \mathbb{Z}^n. \]
Prop 10.40: Let \((\overline{X}, x_0)\) be a path-connected based space. Let \(U, V\) be two open sets s.t. \(U \cap V \ni x_0\) and \(U \cup V\) is path connected. Then the images of \(\pi_1(U, x_0) \to \pi_1(\overline{X}, x_0)\) \& \(\pi_1(V, x_0) \to \pi_1(\overline{X}, x_0)\) are sufficient to generate \(\pi_1(\overline{X}, x_0)\) as a group.

Proof: Let \(\gamma: I \to \overline{X}\) be a path.

We will show \([\gamma] = [\gamma_1] \ast \cdots \ast [\gamma_n]\)
where each \(\gamma_i\) lies entirely in \(U\) or \(V\).

Consider \(\gamma^{-1}(U), \gamma^{-1}(V)\), open subsets of \(I\). We can write each as a union of open intervals in \(I\). Since they cover \(I\) \& \(I\) is compact, we can find a (minimal) finite subcover, writing \(I\) as a union
\[ I = \bigcup (a_i, b_i) \text{ where each interval finite} \]

\[ \text{alternately contained in } \gamma^{-1}(U) \text{ or } \gamma^{-1}(V). \]

Now find \( t_0 < t_1 < t_2 < \ldots < t_m = 1 \)

s.t. \( [t_i, t_{i+1}] \) is contained in either \( \gamma^{-1}(U) \) or \( \gamma^{-1}(V) \), and each \( t_i \in \gamma^{-1}(U \cup V) \)

Now define \( r_i \) as follows

For each \( t_i \), fix a path in \( U \cup V \)
from \( x_0 \) to \( \gamma(t_i) \), denoted \( \sigma_i \).

Then let \( q_i \) denote the path \( \gamma|_{[t_{i-1}, t_i]} \) rescaled to take time 1.
Let $\gamma_i = \sigma_{i-1} \ast \rho_i \ast \hat{\sigma}_i$

then one can easily see that $\gamma_i$ lies entire in $U$ or $V$ if

$$[\gamma_1] \ast [\gamma_2] \ast \ldots \ast [\gamma_m] = [\gamma]$$

in $\pi_1 (\Sigma, x_0)$. \qed

Note: this result relies on the path-connectedness of $U \cup V$ & will fail without it.
Cor 10.42 \[ \text{若干} \] Suppose \( X = U \cup V \) where
\( U, V \), \( U \cap V \) are all path connected & \( U, V \) are simply connected, then \( X \) is simply connected.

Proof: Let \( p, q \) be the north & south poles of \( S^n \) \((0, 0, \ldots, 1)\), \((0, 0, \ldots, -1)\) resp.

\( \text{Proj} : S^n \rightarrow \mathbb{R}^n \)

\[ \text{Proj} \left( x_1, \ldots, x_{n+1} \right) = \frac{1}{1 - x_{n+1}} (x_1, \ldots, x_n) \]

is continuous & has inverse.
$g : \mathbb{R}^n \rightarrow S^n - \{p\}$

$g(y_1, \ldots, y_n) = (t(y) \cdot y_1, \ldots, t(y) \cdot y_n, 1 - t(y))$

where $t(y) = \frac{2}{1 + \|y\|^2}$

$\Rightarrow S^n - \{p\} \approx \mathbb{R}^n \leftarrow s.c.$

$S^n - \{q\} \approx \mathbb{R}^n \leftarrow$

$S^n - \{p, q\} \approx \mathbb{R}^n - \{0\} \leftarrow p.c.$

the previous corollary applies. $\square$
Consequences: \( S' \neq S^n, \ n > 1 \)
\[ S^2 \neq S' \times S' \]

This result...

Example 10.43 | Let \( S' \cup S' \) denote the figure 8.

\[ \pi_1 \left( S' \cup S', o \right) \] contains two elements \( a, b \).

Consider the cover \( T \).
\[ T \xrightarrow{P} S' \cup s' \]

\[ \Pi_{T_1}(T, 0) = \{ e \} \text{ (in fact, } T \text{ is contractible)} \]

\[ (T, o_T) \]

\[ \downarrow P \]

\[ (S' \cup s', o_s) \text{ covering space} \]

A \underline{word} is a list of a's, b's, a^-1, b^-1's in some order.

It is \underline{reduced} if \(a^k a^{-l}, a^r a, b^m b, b^{-n}\) do not \underline{appear}.

\[ \text{E.g. } a a a b a b^{-1} a^{-1} b = a^5 b a b^{-1} a^{-1} b \]

or \(a^5 b^{-19}\)

Any \underline{(reduced) word} gives a path in
& lifts to a path in $T$.

There is a bijection (reduced words)
$\xrightarrow{\text{points in } p^{-1}(O_s)}$.

$= \pi_1(S' \cup S', O_s)$ consists of
the group of (reduced) words in $a, b$
operation is concatenation

\[ a b a^{-1} b^2 \cdot b^{-1} a b^3 = \]
\[ a b a^{-1} b b^{-1} a b^3 = a b a^{-1} b a b^3 \]

The \underline{free} group on two generators.

Observe $ab \neq ba$ (different
points in $p^{-1}(O_s)$ so $\pi_1(S' \cup S')$
is not abelian.