Def 6.1] An open cover of a topological space \( \mathcal{X} \) is a collection \( \{U_i\}_{i \in I} \) of open sets s.t. \( \bigcup_{i \in I} U_i = \mathcal{X} \).

---

Def 6.2] A top. sp. \( \mathcal{X} \) is compact if every open cover \( \{U_i\}_{i \in I} \) contains a finite subcover \( \{U_1, \ldots, U_m\} \subseteq \{U_i\}_{i \in I} \) s.t. \( \bigcup_{i \in I} U_i = \mathcal{X} \).

---

Prop 6.3] Let \( f : \mathcal{X} \to Y \) be a c/k function and let \( A \subseteq \mathcal{X} \) be compact (with the subspace topology). Then \( f(A) = \{ f(a) \mid a \in A \} \).
is compact in $Y$.

**Proof:** Let $U = \{ U_i : i \in I \}$ be an open cover for $f(A)$. Then $\{ f^{-1}(U_i) : i \in I \}$ is an open cover of $A \to f$ has a finite subcover $f(U_1), \ldots, f^{-1}(U_n)$. $\{ U_1, \ldots, U_n \}$ is a finite subcover of $U$.

---

**Prop 6.41** Suppose $X$ is a Hausdorff space and $A \subseteq X$ is a compact subset, then $A$ is closed.

**Proof:** We will show $X \setminus A$ is open. Let $x \in X \setminus A$. For each $y \in A$, choose open sets $U_y, V_y$ s.t. $y \in U_y, x \not\in V_y \Delta U_y = \emptyset$ (Hausdorff property).

The sets $U_y$ form a cover of the compact set $A$, we can find a finite subfamily
\[ y_1, y_2, \ldots, y_n \text{ s.t. } A = \bigcap_{i=1}^{n} U_{y_i}. \text{ Then let } V = \bigcap_{i=1}^{n} V_{y_i} \ni x, V \text{ is an open nbd of } x \in V \cap \left( \bigcup_{i=1}^{n} U_{y_i} \right) = \emptyset, \text{ so } V \cap A = \emptyset. \]

\[ \therefore V \subseteq \bar{\mathbb{X}} \setminus \bar{A} \]
Prove that \( \{U_i\} \) is an open cover of \( A \) (i.e. \( U_i \) open in \( X \), \( \bigcup U_i = A \)).

Then \( \{U_i\} \cup \{X - A\} \) is an open cover of \( X \) and hence a finite subcover \( \{U_1, \ldots, U_n\} \). Remove \( X - A \) from this finite subcover to obtain a finite subcover of the original cover. \( \square \)

**Def 6.6** A topological space \( X \) is sequentially compact if every sequence \( (x_n) \) in \( X \) has a convergent subsequence.
Def 6.7] A metric space \((\mathbb{X}, d)\) is Totally Bounded if \(\forall \varepsilon > 0,\) \(\mathbb{X}\) can be expressed as a finite union of balls \(B(x_i; \varepsilon)\).

Observe: a bounded subset of a totally bounded set is totally bounded.

Easy exercise: totally bounded \(\iff\) bounded.
Prop 6.8) (Borel-Lebesgue)

For a metric space $X$, \textbf{TAKE}:

a) $X$ is compact

b) $X$ is sequentially compact

c) $X$ is complete and totally bounded.

\hline

Proof: a $\Rightarrow$ b) Assume $X$ compact and suppose f.o.c. $(x_n)$ is a sequence with no convergent subsequence. Note that $\{x_n\}_n$ is an infinite set (otherwise say
\[ \text{I would appear as often as giving a convergent subseq.)} \]

Two further things are true:

\[ 1) \quad \forall y \in \mathbb{X} \setminus \{x_n\}, \text{ there exists some } \varepsilon > 0 \text{ s.t. } B(y, \varepsilon) \cap \{x_n\} = \emptyset. \]

(Otherwise, we would have a convergent subseq. This step requires the metric.)

\[ 2) \quad \forall x_i \in \{x_n\}, \exists U_i \text{ open s.t. } U_i \text{ does not contain any other point of } \{x_n\}. \]

Then \( \bigcup U_i \cup (\overline{\mathbb{X}} \setminus \{x_n\}) \) is an open cover with no finite subcover.
b) Sequentially compact $\Rightarrow$ Complete is easy

If \((x_n)\) is Cauchy and has a convergent subsequence

\[ (x_{n_i}) \rightarrow y \]

then \((x_n) \rightarrow y\)

Sequentially compact $\Rightarrow$ totally bounded

Suppose \(X\) is not totally bounded then $\exists \varepsilon > 0$ s.t. $\exists$ is not a finite union of balls $B(x_i, \varepsilon)$

$\Rightarrow$ can find a sequence

\[ (x_1, x_2, ...) \]

s.t. $d(x_i, x_j) > \varepsilon$ $\forall i, j$
this has no Cantor subsequence =
no can. subseq. ☹️

(c) Complete & totally bounded => Compact.

Assume X complete, totally bounded.

Let \{U_i\}_{i \in I} be an open cover and assume f.t.s.o.c. that it has no finite subcover.

We may cover X by finitely many balls of radius \( \frac{1}{\varepsilon} \). At least one of these balls, \( B(x_i, \frac{1}{\varepsilon}) \) requires so many open sets in \( \{U_i\}_{i \in I} \) to cover it (otherwise we could find a finite subcover).
Now covers $B(x_i, \epsilon_i)$ by finitely many balls of radius $1/4_i$, and centres of distance $< \frac{1}{2} + \frac{1}{4_i}$ from $x_i$.

Then at least one of these requires as many elements of $\mathcal{U}_i S$, say $B(x_2, 1/4_i)$.

Proceed like this to form $(x_1, x_2, \ldots)$
s.t. \[ d(x_i, x_{i+1}) < 2^{-i} + 2^{-i-1} \]
and \[ B(x_i, 2^{-i}) \] cannot be covered by finitely many sets in \( \{ U_i \} \).

Then \( (x_i) \) is Cauchy!!

\[
d(x_{i+k}, x_{i}) < \sum_{j=i}^{i+k} \left( 2^{-i} + 2^{-i-1} \right)
= \sum_{j=0}^{k} 2^{-i}(2^{-j} + 2^{-j-1})
< 2^{-i} \sum_{j=0}^{k} 2^{-j+1}
< 2^{-i} \left( 3 \right) = 3 \cdot 2^{-i}
\]

so \( (x_i) \) \( \rightarrow \) \( x \).
Now $x \in U_i$ for some $i \in I$

Moreover, let $\varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq U_i$.

Now find $x_n$ s.t.

1. $d(x_n, x) < \varepsilon/2$
2. $2^{-n} < \varepsilon/2$

Both true by taking $n > 0$

Then $U_i \cap B(x, \varepsilon) \supseteq B(x_n, 2^{-n})$

So $U_i$ is a single element of $\{U_i\}_{i \in I}$ covering $B(x_n, 2^{-n})$

So $\bigcup$ is in fact compact $\mathbb{A}$.
**Corollary 6.9** \( \text{(Heine-Borel)} \)

A subset of \( \mathbb{R}^n \) is compact if

- it is closed and bounded.

**Proof:** For a subset of \( \mathbb{R}^n \)

(or any complete metric space)

- closed.

- all balls in \( \mathbb{R}^n \) are totally bounded.

For \( \varepsilon > 0 \)

Divide \( \mathbb{R}^n \) into \( 2^\varepsilon \) balls where

\[ B((x_1, \ldots, x_n), 2\varepsilon) \quad \text{when} \quad x_i = a_i \varepsilon \quad \text{where} \quad a_i \varepsilon \in \mathbb{Z}. \]

Hence in \( \mathbb{R}^n \), bounded sets are totally bounded.
Prop 6.10 Suppose $\bar{X}$ is compact and $Y$ is Hausdorff and $f: \bar{X} \to Y$ is a continuous bijection. Then $f$ is a homeomorphism.

Proof: As a map of sets, $f$ has an inverse $f^{-1}: Y \to \bar{X}$. We must show that $f^{-1}$ is c.b.s.

We show $(f^{-1})^{-1}(C)$ is closed $\forall$ closed $C \subseteq \bar{X}$.

Let $C \subseteq \bar{X}$ be closed, then $C$ is compact so $(f^{-1})^{-1}(C) = f(C)$ is a compact subset of the Hausdorff space $Y$, so $(f^{-1}(C)$ is closed. $\Box$
Example 6.11 Consider the following map
\[ I \to I/\sim \]
\[ \theta \mapsto (\cos 2\pi \theta, \sin 2\pi \theta) \]
\[ I/\sim \sim \sim \sim S^1 \]
Compact
Quotient topology
Hausdorff

Prop 6.12 A compact Hausdorff space is normal.

Proof: Let \( \tilde{X} \) be compact and Hausdorff.

We first prove \( \tilde{X} \) is regular. Let \( x \in \tilde{X} \)
\( \delta \subset \tilde{X} \) be a point and a closed set
\( \gamma \subset \tilde{X} \). Since \( \tilde{X} \) is c.p.c.t., \( \delta \) is
compact. Since $X$ is Hausdorff, $\forall c \exists x, U_c \ni x$ s.t. $U_c, V_c$ open $U_c \cap V_c = \emptyset$. Then $C \subseteq \bigcup_{c \in C} U_c$

We may replace $\{V_c\}_{c \in C}$ by a finite subcover $\{U_{c_1}, \ldots, U_{c_n}\}$

Then $\bigcap_{i=1}^n U_{c_i}$ is open and disjoint from $U_{c_1} \cup \ldots \cup U_{c_n} \supseteq C$.

The proof of normality is similar. $\Box$. 
Prop 6.14 (Alexander's Subbase Theorem)

Let \( X \) be a topological space and \( \mathcal{B} \) a subbase for \( X \). If every \( \{ \mathcal{B}_i \}_{i \in I} \subseteq \mathcal{B} \) that covers \( X \) has a finite subcover, then \( X \) is compact.

**Proof**

Let \( \mathcal{C} \) denote the set of all open covers of \( X \) having no finite subcover. Suppose \( \text{f.t.s.o.c. } \mathcal{C} \neq \emptyset \), we can find a nonempty \( \mathcal{F} \subseteq \mathcal{C} \) is a chain. Then

\[
F = \bigcup_{i \in I} F_i
\]

is an open cover. Any finite...
subcover would lie in some $F_i$, so $F$ has no finite subcover. Hence $F$ is an upper bound in $C$. So $\exists C$ contains a maximal element $F_{\text{max}}$. Now take $F_{\text{max}} \cap B$. Is this still an open cover? Suppose not. Let $x \notin U \cup U$. Then $x \notin F_{\text{max}} \cap B$.

But $x \in U \cup U$, so $x \notin F_{\text{max}} \cup F_{\text{max}}$

$x \in U_x$. Write $U_x = B_1 \cap \cdots \cap B_n$ each in $\mathcal{B}_1$ \space name in $F_{\text{max}}$.

All of the sets $F_{\text{max}} \cup \{B_i \cap \cdots \cap B_n \}$
must admit finite subcovers

So we can write

\[ \overline{X} = \bigcup_{i} V_i \cup B_i \]

\( = \) finite union of sets in \( \mathcal{F}_{\text{max}} \)

but now

\[ \overline{X} = \bigcup_{i} V_i \cup U_2 \]

\( = \) finite union of sets in \( \mathcal{F}_{\text{max}} \)

So \( \mathcal{F}_{\text{max}} \cap B \) must be a cover consisting of subbasis cells.

but it must now admit a finite subcover.

So \( C \) must be empty.  \( \Box \)
Theorem 6.15 (Tychonoff Theorem)

Let \( \{X_i : i \in I\} \) be a family of compact spaces. Then \( \prod_{i \in I} X_i \) is compact.

Proof: It suffices to prove that any cover
\[
U = \left\{ \prod_{i \in I} U_i : i \in I, U_i \text{ open in } X_i \right\}
\]
has a finite subcover.

For any \( i \in I \) define
\[
U_i = \left\{ U \mid \pi_i^{-1}(U) \subseteq U, U \text{ open in } X_i \right\}
\]
We claim that for at least one \( i \), the set \( U_i \) is an open cover for \( X_i \).

If not, for each \( i \), we find \( x_i \in X_i - \bigcup_{i \in I} U_i \).
and the $(x_i)$ assemble to make $x \in \mathbb{X}$

which isn't in $\pi_i^{-1}(U)$ for any such $\pi_i$ in $\mathbb{U}$.

So for some $i$, $U_i$ is a cover for $\mathbb{X}$.

It has a finite subcover $U_1, \ldots, U_n$

then $\pi_i^{-1}(U_1) \cap \cdots \cap \pi_i(U_n)$ is a finite subcover of $\mathbb{X}$.

Small observation:

**Prop 6.16** The union of finitely many compact subsets $K_1, \ldots, K_n$ is compact.

Proof: Sufficient by induction to do
Suppose \( U \) is a cover of \( K_1 \cup K_2 \) then we can find \( U_1, \ldots, U_n \) covering \( K_1 \) and \( U_{n+1}, \ldots, U_{n+m} \) covering \( K_2 \).

Compactification

**Def 6.17**: let \( X \) be a topological space. A compactification of \( X \) is a map \( i : \overline{X} \rightarrow K \) satisfying

1. \( \overline{X} \) is homeomorphic to \( \text{Im} \ i \) (i.e. \( i \) is an embedding)

2. \( \text{Im} \ i \subseteq K \) is dense (i.e. \( \overline{\text{Im} \ i} = K \)).

3. \( K \) is compact.

These are most easily studied when both spaces
are Hausdorff.

Prop 6.18: If $X$ is compact & $i: X \rightarrow K$ is a Hausdorff compactification, then $i$ is a homeomorphism.

Proof: $i$ is compact in a Hausdorff space, therefore closed, therefore $\text{Im } i = K$. A continuous bijection with compact source & Hausdorff target is a homeomorphism.

Example 6.19: A noncompact Hausdorff space may have several non-homeomorphic compactifications.

Consider $(0, 2\pi)$ with the usual topology. Clearly the inclusion $(0, 2\pi) \subseteq [0, 2\pi]$ is a compactification.

Now take $f: (0, 2\pi) \rightarrow S^1$ given by $f(\theta) = (\cos \theta, \sin \theta)$.

We know $S^1$ is compact -- for instance it is closed & bounded.

$\text{Im } f = S^1 \setminus \{(1, 0)\}$, an open subset of $S^1$ we know $f$ is an open, continuous function.

Hence $f$ is a homeomorphism onto its image $\text{Im } f = S^1$, e.g. by considering a sequence.
**Theorem 6.20**: Suppose $(\overline{X}, \tau)$ is a non-compact space. Form a set $K = \overline{X} \cup \{\text{new element}\}$ and give it the topology with basis

1. $U \in \tau$ (U open in $\overline{X}$)

2. $V_C = (\overline{X} \cup \{\text{new element}\}) \setminus C$ where $C \subseteq \overline{X}$ is closed and compact in $\overline{X}$.

We call this space $K$ the one-point compactification of $\overline{X}$.

That this indeed is a basis for a topology is left as an exercise — closed subsets of $\overline{X}$ are closed and compact.

**Prop 6.21**: The one-point compactification is a compactification.

**Proof**: View $\overline{X} \subseteq K$. The sets $V_C \cap \overline{X}$ with $V_C$ as in 6.20 are all of the form $\overline{X} \setminus C$ where $C$ is closed and compact. It follows $V_C \cap \overline{X}$ is open in $\overline{X}$. So too is $\cup V_C$. It follows the subspace topology on $\overline{X} \subseteq K$ is the topology $\tau$ (of our bigger $\overline{X}$). So $i: \overline{X} \rightarrow K$ is an embedding.
Since $\overline{X}$ is not compact, $K \setminus \overline{X}$ is not open in $K$, so $\overline{X}$ is not closed in $K$ so $\overline{X} = \overline{X} \cup \{\infty\} = K$. ②

Suppose $U = \{U_i : i \in I \cup \{V_C\} \}$ is an open cover of $K$ by basis sets. To show $K$ is compact, it suffices to show any such cover has a finite subcover.

Note that $I \neq \emptyset$. Let $V_C \in \infty$ be in the cover. Then $\overline{X} \setminus V_C$ is covered by $U$ as well, so $\overline{X} \setminus V_C$ is closed & compact.

So $\exists \{U_1, \ldots, U_n, V_C, \ldots, V_n\}$ covering $\overline{X} \setminus V_C = \{U_1, \ldots, U_n, \ldots, V_n\}_{n \in \mathbb{N}}$ is a finite subcover of $U$. ③
Def 6.22 | We say a space $\mathcal{X}$ is locally compact if $\forall x \in \mathcal{X} \exists$ an open $U$ and compact $C \subseteq U$ with $C$ compact. (we call $C$ a compact n'bd)

Prop 6.23 | If $\mathcal{X}$ is Hausdorff TFAE
1. $\mathcal{X}$ is locally compact
2. every open n'bd $x \in U$, $\exists$ a compact n'bd $C \ni x$ with $C \subseteq U$.

Proof: $2 \implies 1$ is trivial

$\implies 2$. Suppose $\mathcal{X}$ is locally compact. Let $x \in U$ be a point and an open n'bd.

$x$ has a compact n'bd $B$

$x \in V \subseteq B \land B \setminus V$ is closed, disjoint from $x$.

Since $B$ is compact Hausdorff, it is regular, so we can find an open set $w \subseteq B$ (in the subspace topology on $B$) s.t.

$x \in w \subseteq w = U \cap B \subseteq B$

$w$ is a closed subset of $B$, so is a closed subset of $\mathcal{X}$. It is compact, and it contains $(w \cap \mathcal{X} \setminus B) \cap V \ni x$, an open n'bd of $x$. $\Box$
Example 6.24) In $(\mathbb{R}^n, d_\infty)$, any ball $B_p(\bar{x}, \varepsilon)$ has closure $\bar{B}_p(\bar{x}, \varepsilon)$

$$\{ y \in \mathbb{R}^n : d_p(\bar{x}, y) \leq \varepsilon \}$$

which is closed and bounded in $\mathbb{R}^n$. So $\mathbb{R}^n$ is locally compact.

In $\mathbb{Q}$ (with the metric topology), no interval $[a, b] \cap \mathbb{Q}$ is compact. Choose a sequence in $[a, b] \cap \mathbb{Q}$ converging to an irrational number in $\mathbb{R}$. Therefore $(a, b) \cap \mathbb{Q}$ is not contained in any compact subset of $\mathbb{Q}$, so $\mathbb{Q}$ is not locally compact.

Prop 6.25) Suppose $\mathbb{X}$ is Hausdorff, locally compact but not compact. Then the one-point compactification $K$ of $\mathbb{X}$ is Hausdorff.

Proof: $K = \mathbb{X} \cup \{ \infty \}$ as sets. $\mathbb{X}$ is Hausdorff per se. The only thing to check is that if $x \in \mathbb{X}$, then $U$ an open $U \ni x \& \exists{\varepsilon} > 0 \text{ s.t. } U \cap V = \emptyset$. 
but if \( C \) is a compact nbhd of \( x \) then \( x \in U \subseteq C \) & \( x \in K \setminus C \) works.

**Prop 6.26:** A locally compact Hausdorff space \( X \) is regular.

**Proof:** \( X \) is a subspace of a compact Hausdorff space, the 1 point cichlidhe. \( D \)

**Prop 6.27:** If \( A \subseteq X \) is a subspace of a locally compact Hausdorff space, & \( A \) is either open or closed, then \( A \) is locally compact.

**Proof:** (open case). \( X \) is normal. We may find a compact nbhd \( C \) of \( x \), \( x \in U \subseteq C \) open \( \Rightarrow \) compact.

\[
A \cap U \text{ is open in } X, \ x \in A \cap U
\]
by normality, can find some open $V \subseteq \mathbb{R}$ s.t. $x \in V \subseteq \overline{V} \subseteq A \cap U$.

Then $x \in \overline{V}$ is a closed subset of $C$.

$\implies \overline{V}$ is compact. $V \subseteq \overline{V} \subseteq A$ as so that $\overline{V}$ is a compact n'bd of $x$ in $A$.

Closed case

Let $C \ni x$ be a compact n'bd of $x$ in $A$ & $C \not\ni 2x$.

Then $C \cap A$ is compact & closed in $A$. $U \cap A$ is open in $A$.

$x \in U \cap A \subseteq C \cap A$ \quad \Box
Thm 7.21) A open in locally cpt.

\[ \overline{\text{defn}} \quad X = A \cup \text{cpt tedoff}. \]

(the same is true if \( A \) closed)

Proof.

\[ A \]

\[ \text{defn} \]

\[ A \subset \overline{\mathcal{X}} \subset \mathcal{Y} \]

one point compactification
$\mathcal{U}$ is equidense if $\mathcal{U}$ is regular.

$Y - A$ is closed.

$\varepsilon \in Y - A \Rightarrow \exists U \in \mathcal{U}, \forall x \in U \cap A$

$U \cap V = \emptyset$

$\overline{x} \in U \cup A \subset \overline{V} \subset A$

Closed in $Y$, so compact.

Continuous in $A$, so closed in $A$.

$x \in U \cup \overline{U} \subset A$