5. Sequences

**Def 5.1**

A sequence \((x_n)_{n=1}^\infty\) in a space \(X\) is a function \(x : \mathbb{N} \to X\).

Given \(\mathbb{N}\) the discrete topology, then \(x\) is continuous.

**Def 5.2**

A sequence \((x_n)_{n=1}^\infty\) converges to \(y \in X\) if, for every open \(U \ni y\), there exists \(N \in \mathbb{N}\) such that \((x_n)_{n=N}^\infty \subseteq U\).

**Def 5.3**

Let \(\tilde{\mathbb{N}}\) be the space \(\mathbb{N} \cup \{\infty\}\) given the following topology:

\(U \subseteq \tilde{\mathbb{N}}\) is open if \(U \not\ni \infty\) or \(U = \tilde{\mathbb{N}}\).
if \( u \neq 0 \) and \( n > 0 \), s.t. \( n > N \) for \( n \in U \).

Closed sets are either

1. finite
2. compact.

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Prop 5.41 A sequence \( x_n \) converges to \( y \in \overline{X} \) if and only if the function

\( \tilde{x} : \mathbb{N} \to \overline{X} \) given by

\[
\tilde{x}(n) = x_n \quad \text{and} \quad \tilde{x}(\infty) = y
\]

\( \mathbb{N} \to \overline{X} \) with

\[
\tilde{x} : \mathbb{N} \to \overline{X} \quad \text{given by}
\]

\[
\tilde{x}(n) = x_n \quad \text{and} \quad \tilde{x}(\infty) = y
\]

\( \mathbb{N} \to \overline{X} \) with
is continuous.

Proof: Suppose this function is continuous then for any open \( y \in \tilde{x}^{-1}(U) \) is open \& contains \( \infty \Rightarrow \exists N > 0 \)

s.t. \( n \in \tilde{x}^{-1}(U) \) \& \( n > N \Rightarrow \)

\( \tilde{x}(n) = x_n \in U. \)

Conversely, suppose the sequence \( x_n \) converges \( y \). Let \( U \) be open in \( \tilde{x} \). then either \( y \notin U \Rightarrow \tilde{x}^{-1}(U) \neq \infty \)

\( \Rightarrow \tilde{x}^{-1}(U) \) is open or

\( y \in U \Rightarrow \) some ball of \( x_n \in U \)

\( \Rightarrow \tilde{x}^{-1}(U) \) is open in \( \tilde{N}. \)
Prop 5.5: Suppose \((x_n)\) is a sequence in \(X\), converging to \(y\). Suppose \(f: X \rightarrow \mathbb{R}\) is a continuous function. Then \(f(x_n)\) converges to \(f(y)\).

Proof:

\[
\begin{array}{c}
N \\ \downarrow \\
\tilde{x} \\ \downarrow \\
\tilde{x} \\
\end{array} \quad \begin{array}{c}
\tilde{x} \\ \downarrow \\
f(\tilde{x}) \quad \rightarrow \\
\end{array} \quad \begin{array}{c}
\tilde{y} \\ \downarrow \\
f(\tilde{y}) \quad \rightarrow \\
\end{array}
\]

Prop 5.6: Suppose \((x_n)\) is a sequence in \(X\), converging to \(y\). Suppose \((x_{n_m})\) is a subsequence. Then \((x_{n_m}) \rightarrow y\)
Proof: \((\tau_{n_m})_m\) is the composite

\[ \begin{array}{ccc} N & \xrightarrow{\tau} & N \\ \downarrow & & \downarrow \\ \tau_n & \xrightarrow{x} & \tau_x \end{array} \]

\(m \mapsto x(\tau_n(m)) = x_{\tau_n} \).

We extend \(\tau\) to \(\tilde{\tau}\) by \(\tilde{\tau}(\infty) = \infty\). Then \(\tilde{\tau} : N \to \bar{N}\) is continuous; if \(\bar{U}\) is open either:

a) \(\tilde{\tau}^{-1}(U) = \{m \in N \mid \tau_n(m) \in U\}\)

b) \(\exists N \text{ s.t. } \{m \mid m > N\} \not\subseteq \bar{U} \Rightarrow \{m \mid m > N\} \subseteq \tilde{\tau}^{-1}(U)\).

hence the extension

\[ \begin{array}{ccc} N & \xrightarrow{\tilde{\tau}} & \bar{N} \\ \downarrow & & \downarrow \\ \bar{N} & \xrightarrow{x} & \bar{X} \end{array} \]

which sends \(\{\infty\} \to y\).

\(\tilde{\tau}\) is continuous. \(\blacksquare\)
Prop 5.7: Suppose $A \subseteq X$ is a closed subset of a top. space. Let $(x_n)$ be a sequence with no many terms in $A$, converging to $y \in X$. 

Sequences and closure
Then \( y \in A \).

Proof

\[
\begin{array}{ccc}
\mathbb{N} & \xrightarrow{\pi} & A \\
\downarrow & & \downarrow \text{inclusion of a closed set} \\
\mathbb{N} & \xrightarrow{\bar{x}} & \bar{x} \\
\end{array}
\]

\( \bar{x}^{-1}(A) \) is closed

Therefore, \( \bar{x} \) is continuous. Many values of \( n \in \mathbb{N} \)

What sets are closed in \( \bar{\mathbb{N}} \)? Either they contain \( \infty \) or they are finite.

\( \Rightarrow \) \( \bar{x}^{-1}(\{\infty\}) = \infty \Rightarrow \bar{x}(\infty) = y \in A. \)
Def 5.8) Given $A \subseteq \overline{X}$, the sequential closure $\overline{A}^{\text{seq}}$ of $A$ is the set of all limits of sequences contained in $A$.

\[ \overline{A}^{\text{seq}} \subseteq \overline{A} \] \hspace{1cm} \text{hence say } A \text{ is sequentially closed if } \overline{A}^{\text{seq}} = A

Prop 5.9) If $X$ is first countable and $A$ is a sequentially closed subset, then $A$ is closed.

Let $a \in \overline{A}$. Let \( \{U_1, U_2, \ldots \} \) be a countable basis for the topology at $a$ and let \( U_i' = \bigcap_{j=1}^{i} U_i \), so...
$U_1 \supset U_2 \supset U_3 \cdots$

Since $a \in A$ and $U_i$ is open around $a$, we must have $U_i \cap A \neq \emptyset$
so we can find $x_i \in A \cap U_i$.

Then for any open $V \ni a$, I can
$a \in U_i \subset V$
and $\exists (x_i, x_{i+1}, \ldots)$ is
a tail of $(x_n)_{n \in \mathbb{N}}$ in $U$. So
$(x_n)_{n \in \mathbb{N}} \rightarrow a$.

So $a \in A$. 

$U_1 \supset U_2 \supset \ldots \supset U_i$, $A$
Prop 5.10) Suppose \( X, Y \) are topological spaces, \( X \) is first countable and \( f : X \to Y \) is a function such that \( (x_n)_n \to x \)
\[ \implies (f(x_n))_n \to f(x) \]
Then \( f \) is continuous.

Proof: It suffices to prove \( f^{-1}(C) \) is closed in \( X \) for all closed \( C \subseteq Y \).

Let \( C \) be closed in \( Y \). Let \( x \in \overline{f^{-1}(C)} \) and choose a sequence \( (x_n)_n \to x \) with \( x_n \in f^{-1}(C) \)
Then \( (f(x_n))_n \to f(x) \)
\[ \in C \]
\[ \implies x \in f^{-1}(C) \] So \( f^{-1}(C) \) is closed.
Prop 5.11 Suppose \((x_n)_{n \in \mathbb{N}} \rightarrow y_1\) and \((x_n)_{n \in \mathbb{N}} \rightarrow y_2 \text{ in } \overline{X}\). If \(\overline{X}\) is Hausdorff then \(y_1 = y_2\).

If \(y_1 \neq y_2\), then we can create disjoint open subsets \(U \ni y_1, \overline{U} \ni y_2\). \(U\) cannot contain a tail of the sequence \((x_n)_{n \in \mathbb{N}}\) so \((x_n)_{n \in \mathbb{N}} \rightarrow y_1\).

Example 5.12) The line with doubled origin: \(\mathbb{R} \cup \{0^*\}\) be given the following topology.
\( (a, b) \) open intervals are open

\( (a, b) - \{0 \text{ or } \infty\} \) are open when \( a < 0 < b \)

![Diagram]

Then this space is \( T_1 \) (disregarding \( 0, \infty \) it's Hausdorff)

\((-1, 1) \cap (-1, 1) = \{0\} \cup \{\infty\}\)

but \( (\frac{1}{n})_{n=1}^\infty \rightarrow 0 \)

and \( (\frac{1}{n})_{n=1}^\infty \rightarrow \infty \).

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**Prop 5.13** Suppose \( \prod_{i \in I} X_i \)
is a product of topological spaces, and 
\[(x_n) \subseteq X \text{ is a sequence. Then} \]
\[(x_n) \xrightarrow{\text{y}} \text{ if and only if} \]
\[\left(\pi_i(x_n)\right)_n \xrightarrow{\text{y}} \pi_i(y) \quad \forall i \in I.\]

**Proof:** \(\pi_i\) is continuous \(\forall i\), so 
\[(x_n) \xrightarrow{\text{y}} \Rightarrow \left(\pi_i(x_n)\right)_n \xrightarrow{\text{y}} \pi_i(y).\]

Conversely, suppose \(\left(\pi_i(x_n)\right)_n \xrightarrow{\text{y}} \pi_i(y) \quad \forall i \in I,\) then let \(U \ni y\) be an open set. \(U\) contains a basis element of the form 
\[U \supseteq \pi_{i_1}^{-1}(U_{i_1}) \cap \cdots \cap \pi_{i_k}^{-1}(U_{i_k}) \ni y\]
where each \(i_j \in I \text{ and } U_j \text{ open in } X_{i_j} \).

There for each \(j \in \{1, \ldots, k\}, \exists N_j > 0\)
s.t. \( m > N_j \Rightarrow \pi_i(x_m) \in U_j \)

Take \( N = \max_j \{ N_j \} \). Then

\[ m > N \Rightarrow \pi_{ij}(x_m) \in U_j \quad \forall j \]

which implies

\[ (x_m) \in \bigcap_j \pi_{ij}^{-1}(U_j) \subset U \]

and so the arbitrary nbhd of \( y \) contains a tail of the sequence, so \( (x_m) \to y \). \( \Box \)
Cauchy Sequences

Defn 5.14) Let \((X,d)\) be a metric space, and \((x_n)_{n=1}^{\infty}\) a sequence. We say \((x_n)_{n=1}^{\infty}\) is Cauchy if \(\forall \varepsilon > 0 \exists N > 0 \text{ s.t. } d(x_n, x_m) < \varepsilon \text{ for } n, m > N.

Observations 5.15) This defn requires a metric.

1. If \((x_n) \rightarrow y\) converges, then \(\forall \varepsilon > 0 \exists N \text{ s.t. } d(x_n, y) < \varepsilon/2 \text{ (i.e. } x_n \in B(y, \varepsilon))\).
For \( n > N \), then
\[
d(x_n, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
for \( n, m > N \).

So Convergent \( \Rightarrow \) Cauchy.

\[\text{Def. } 5.16\) A metric space \( \mathbb{X} \) is complete if every Cauchy sequence converges to a limit in \( \mathbb{X} \).\]
Homeomorphic topological spaces may have different Cauchy sequences.

Example 5.17 | The metric space $\mathbb{R}$ is complete; c.f. real analysis.
Example 5.18] Let \( p \in [1, \infty] \) and let \( n \in \mathbb{N} \). Then \( \mathbb{R}^n \) with the \( d_p \) metric is complete.

We do the case \( p < \infty \). Suppose \( (\bar{x}_m)_m \) is a Cauchy sequence

\[
\bar{x}_m = (x_{m,1}, x_{m,2}, \ldots, x_{m,n})
\]

Since the sequence \( (\bar{x}_m)_m \) is Cauchy for any \( \varepsilon > 0 \), \( \exists \ N = N_{\varepsilon} \) s.t.

\[
\left( \sum_{i=1}^{n} |x_{m,i} - x_{m',i}|^p \right)^{\frac{1}{p}} < \varepsilon
\]

whenever \( m, m' > N \).

\[
\Rightarrow \sum_{i=1}^{n} |x_{m,i} - x_{m',i}|^p < \varepsilon^p
\]
\[ \Rightarrow \| x_m, i - x_n, i \| < \varepsilon \]

which is to say that \((x_m, i)_m\)

is a Cauchy sequence in \(\mathbb{R}^n\)

hence \((x_m, i)_m \rightarrow x_i\) for some

\(x_i\). Writing \(\bar{x}\) for the element

\(\bar{x} = (x_1, \ldots, x_n)\), we see by

Prop 5.13 that \(\bar{x}_m \rightarrow \bar{x}\) componentwise

\[ \Rightarrow \bar{x}_m \rightarrow \bar{x} \text{ in the product topology} \]

\[ \Rightarrow \bar{x}_m \rightarrow \bar{x} \text{ in } (\mathbb{R}^n, d_\infty) \]

\[ = (\mathbb{R}^n, d_p) . \]

The case \(p = \infty\) is similar.

The case of \(l^p\) spaces also
Observation: a subspace \( A \subseteq \mathbb{R}^n \) (with any \( d_p \) metric) is complete if and only if it is closed.

Given a canonically sequence in \( A \), a limit exists in \( \mathbb{R}^n \). The set of all limits of sequences in \( A \) is \( \overline{A} \).