Defn 4.1. Let \( X \) be a topological space. If, for any pair of points \( x, y \) in \( X \), there exists

a) an open set \( U \ni x \), \( U \ni y \) or
   \( U \ni x \), \( U \ni y \)

then \( X \) is \( T_0 \).

b) an open set \( U \ni x \), \( U \ni y \)

then \( X \) is \( T_1 \).

c) two open sets \( U \ni x \), \( U \ni y \)

\( U \cap U' = \emptyset \)

then \( X \) is Hausdorff or \( T_2 \).

Hausdorff \( \Rightarrow T_1 \Rightarrow T_0 \).
Prop 4.2: Any metric topology is Hausdorff.

Proof: Given $x, y \in \mathcal{X}$, let $d(x, y)$, define $\varepsilon = \frac{1}{2} d(x, y)$. Then $B(x; \varepsilon) \cap B(y; \varepsilon) = \emptyset$.

For any such $\varepsilon$ satisfies

$$d(x, z) < \varepsilon, \quad d(y, z) < \varepsilon$$

$$d(x, y) = 2 \varepsilon + \epsilon$$

so $B(x; \varepsilon), B(y; \varepsilon)$ form the required pair of open sets.

Example 4.3: The following spaces are not Hausdorff and therefore not metrizable.
(a) Any set with at least 2 elements
of the indiscrete topology.

(b) Any infinite set with the cofinite topology (finite sets "& \{\emptyset\} are closed.)

(c) The 2 element set with open sets

\[ \cdot P_1 \quad \cdot P_2 \]

as indicated

This is also \( I/(0,1) \).

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Example (a) is not \( T_0 \). Example
(b) is \( T_1 \) but not Hausdorff (\( T_2 \)).

Example (c) is \( T_0 \) but not \( T_1 \).

(Any open nbhd of \( P_1 \) contains \( P_2 \).)
The Countability Axiom

Def 4.4.1 If $(X,\tau)$ is a topological space & $x \in \overline{X}$, then a local basis $U$ for $T$ at $x$ is a set $U \in \tau$ such that:
- $x \in U, A \subseteq U$
- $A$ open $\forall x, \exists U \in U$ s.t. $x \in U \subseteq V$. 
Def 4.5] If \( X \) is a topological space, we say \( X \) is first countable if every \( x \in X \) has a countable local basis.

Prop 4.6] Every metric topology is first countable.

Proof: Suppose \((X,d)\) is a metric space; let \( x \in X \). Consider \( F = \{ B(x, \frac{1}{n}) \}_{n=1}^{\infty} \). Let \( U \ni x \) be open, then \( \exists \varepsilon > 0 \) s.t. \( B(x, \varepsilon) \subseteq U \)

Choose \( n > \frac{1}{\varepsilon} \) then \( B(x, \frac{1}{n}) \subseteq B(x, \varepsilon) \subseteq U \), so \( F \) is a local basis at \( x \). \( \Box \)
Def 4.7 A topological space $X$ is second countable if it has a countable basis.

Note second countable $\Rightarrow$ first countable.

Example 4.8 $\mathbb{R}$ with the usual topology is second countable. Given any $x \in \mathbb{R}$ if $(a, b) \ni x$, we can find rational numbers $p, q$ in $\mathbb{Q}$ such that $a < p < x < q < b$.

In particular, if $\mathcal{F} = \{ (p, q) \}_{p \in \mathbb{Q}, q \in \mathbb{Q}}$ then $\mathcal{F}$ is countable if for any
open $U$ if $x \in U \Rightarrow (p, q) \in U$

Prop 4.9 A space with a countable subbasis is second countable.

Proof Let $\mathcal{B}$ be a countable subbasis for $\mathcal{B}$. Let $B_j$ denote the set of intersections of $j$-tuples of sets in $\mathcal{B}$. Then $\mathcal{B}^j \rightarrow B_j$, so $B_j$ is countable. $\mathcal{B} = \bigcup_{j=1}^{\infty} B_j$ is therefore a countable basis. $\square$
Prop 4.10: Let \( \{ X_i \}_{i \in \mathbb{Z}} \) be a countable set of second countable spaces, let 
\( \bar{X} = \prod_{i \in I} X_i \) be given the product topology. Then \( \bar{X} \) is second countable.

Proof: Let \( \mathcal{B}_i \) denote a countable basis for \( \bar{X}_i \). Then \( \mathcal{B}_i = \{ \pi_i^{-1}(U) : U \in \mathcal{B}_i \} \) is a countable set.

\[ S = \bigcup_{i \in I} V_i \] is a countable set.

\( S \) is a subbasis for the product topology.

(For any open \( V \subseteq \bar{X} \), \( \pi_i^{-1}(V) \) is expressible as \( \bigcup_{U \in \mathcal{B}_i} \pi_i^{-1}(U) \) in \( \mathcal{B}_i \).)
so one can generate the usual subbasis using this one).

**Corollary 4.11** \( \mathbb{R}^n \) with the usual topology is second countable.

Alternative proof: verify that the set 
\[
\left\{ B_2(\bar{x}, r) \mid \bar{x} = (x_1, \ldots, x_n) \in \mathbb{Q}^n, r \in \mathbb{Q} \right\}
\]
forms a countable basis for \( \mathbb{R}^n \).

**Defn 4.12** A subset \( A \subseteq \mathbb{R} \) is said to be dense if \( \overline{A} = \mathbb{R} \).

Equivalently, \( A \) is dense if every open set \( U \subseteq \mathbb{R} \) meets \( A \) (\( U \cap A \neq \emptyset \)).
Defn 4.13 | A subset $A \subseteq \mathbb{R}$ is **sparse** if $\overline{\mathbb{R}} - \overline{A}$ is dense.

$A$ is sparse $\Rightarrow \overline{\mathbb{R}} - \overline{A} = \mathbb{R}$

$\Rightarrow$ no nonempty open set contained in $\overline{A}$

$\Rightarrow$ $\text{Int}(\overline{A}) = \emptyset$.

---

Example 4.14 | $\mathbb{Q} \subseteq \mathbb{R}$

$I = \mathbb{R} - \mathbb{Q}$ (irrational numbers)

both $\mathbb{Q}$, $I$ are dense in $\mathbb{R}$.

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Defn 4.15 | A topological space is **separable** if it contains a dense open set.

A second countable space is separable.
Prop 4.165
A separable metric space is second countable.

Choose a countable dense set \( S \subseteq \mathbb{R} \) and let \( \mathcal{B} = \{ B(x, \frac{1}{n}) \mid x \in S, n \in \mathbb{N} \} \).

Let \( U \subseteq \mathbb{R} \) be an open nbhd of a point in \( \mathbb{R} \). Then \( B(y, 4\varepsilon) \subseteq U \), for some \( \varepsilon > 0 \).

But now we can find \( x \in S \) s.t. \( \frac{1}{2} > \varepsilon \). Then \( x \in B(y, \varepsilon) \subseteq B(y, 4\varepsilon) \).

So we can find \( n \) s.t. \( \frac{1}{n} < 2\varepsilon \) and so \( B(x, \frac{1}{n}) \subseteq B(y, 4\varepsilon) \subseteq U \).

\[ y \in B(x, \frac{1}{n}). \]

This is covered in Munkres' 3.80, along
with Lindelöf. We ignore the Lindelöf property.

Example 4.17) Note

First Countable \implies \text{ Separable}

Second Countable \implies \text{ Separable}

\text{Metrizable } \implies \text{ Separable}

\text{If metrizable } \implies \text{ Separable } \implies \text{ Second countable.}

Give any uncountable set the discrete metric, then the result is metrizable \& first cible but not second countable.

Example 4.18) Let \( \mathbb{R} \) be given the half-open interval topology, so
sets of the form $[a, b)$ form a basis. Let $U$ be an open set containing $x$; then $[x, x + \varepsilon) \subseteq U$ for some $\varepsilon > 0$.

$\Rightarrow \quad [x, x + h) \subseteq U$ for some $n \in \mathbb{N}$.

$\mathcal{B}_x = \{[x, x + h) \mid n \in \mathbb{N}\}$ is a countable local basis for the topology at $x$.

$\mathbb{R}_v$ is first countable.

Any open set contains $[a, b)$, which contains some $q \in \mathbb{Q}$, so $\mathbb{Q} \subseteq \mathbb{R}$ is a countable dense subset. $\mathbb{R}_v$ is separable.
This space is not second countable.

Suppose \( B \) is a basis of \( x \in \mathbb{R} \). Then any open interval containing \( x \) in \([x, a)\) must contain an open interval of form \([x, a)\). Therefore there is an injective map \( \mathbb{R} \to B \), and \( B \) is not countable.

---

**Regularity of Normality**

**Def 10.18:** Let \( X \) be a space such that points of \( x \in X \) are closed. We say \( X \) is

a) regular if \( A \cap \overline{X} \) and all closed sets \( C \subseteq X - \{x\} \), there
exists open sets $U \ni x, \quad U \subset C$
such that $U \cap V = \emptyset$.

b) normal if $\forall$ closed sets $A, C$

$s.t. A \cap C \neq \emptyset$, $\exists$ open sets $U, V$

$U \supset A, \quad V \subset C, \quad U \cap V = \emptyset$

Alternate def: (Munkres 31.1)

• $X$ is regular if $A \subset X \\& \ U \subset X$ open
a.) \( \overline{V} \subseteq u \)

- If normal, if \( A \) closed \( C \) & \( u \supseteq C \) open, \( \uparrow \) \( V \supseteq u \) open
- \( \overline{V} \subseteq u \).

________

\[ T_0 \leq T_1 \leq \text{Hausdorff} \leq \text{Regular} \leq \text{Normal} \]

Prop 4.19 Every metrizable space is Normal
Proof: Let \((X, d)\) be a metric space. Let \(A, C\) be disjoint closed sets. Let \(a \in A\) be a point.

\(a \in X - C\), which is open, so \(\exists \varepsilon_a > 0\) \(\forall x \in B(a, \varepsilon_a) \cap C = \emptyset\).

Similarly we find balls \(B(c, \varepsilon_c) \cap A = \emptyset\)
around points in C.

Claim \(\forall a,c, B(a, \varepsilon_a/e) \cap B(c, \varepsilon_c) = \emptyset\); otherwise

\(d(a, c) < \frac{\varepsilon_a}{2} + \frac{\varepsilon_c}{2} < 2 \max\{\frac{\varepsilon_a}{2}, \frac{\varepsilon_c}{2}\}\)

\(\Rightarrow B(a, \varepsilon_a) \cap C \neq \emptyset\) or \(\emptyset\).
Now take \( \bigcup_{a \in A} B(a, \varepsilon/2) = U \)
\[
\bigcup_{c \in C} B(c, \varepsilon/2) = V. \quad \Box
\]

---

**Prop 4.20:** A regular, second countable space is normal

Let \( \overline{X} \) be a regular space and let \( B \) be a countable basis for \( \overline{X} \).

Let \( A, C \) be disjoint closed sets in \( \overline{X} \).

For each \( a \in A \) we can find \( U \subseteq B \), \( U \ni a \) s.t. \( \overline{U} \cap C = \emptyset \)
So we can find a countable family
\[ \{ U_i : i = 1, 2, \ldots \} \]
\[ \text{s.t. } A \subseteq \bigcup_{i=\infty} U_i \]
\[ \overline{U_i} \cap C = \emptyset. \]
Similarly, find \[ \{ V_i : i = 0, -1, \ldots \} \]
\[ \text{s.t. } C \subseteq \bigcup_{i=1} V_i \]
\overline{V_i} \cap A = \emptyset.

Now replace \( U_i \) by \( U_i' = U_i - \bigcup_{j=1}^{i} \overline{V_j} \)

\( V_i \) by \( V_i' = V_i - \bigcup_{j=i+1}^{\infty} \overline{U_j} \).

\( U_i', V_i' \) are open.

\( A \subseteq \bigcup_{i=1}^{\infty} U_i' \), \( C \subseteq \bigcup_{i=1}^{\infty} V_i' \).

Consider \( U_i' \cap V_j' \). w.l.o.g. \( i \leq j \)

then \( V_j' \cap U_i' \leq V_j' \cap U_i \)

\( \leq V_j' \cap \overline{U_i} = \emptyset \).

So \( \bigcup_{i=1}^{\infty} U_i' \cap \bigcup_{i=1}^{\infty} V_j' = \emptyset \).
so \( \bigcap_{i=1}^{\infty} u_i \), \( \bigcap_{i=1}^{\infty} v_i \) are disjoint open sets. \( \Box \)

Prop 4.21 (Munkres 31.2) A subspace (resp. product) of Hausdorff (resp. regular) spaces is Hausdorff (resp. regular).

Proof. These are mainly ‘easy’ to prove. We will prove a product of regular spaces is regular.

Suppose \( X = \prod_{i \in I} X_i \) is a product
of regular spaces, \( \bar{X} \). If \( A_i \) are subsets of the \( \bar{X}_i \), we write \( \bigcap_{i \in I} A_i \) for

\[
\bigcap_{i \in I} \pi_i^{-1}(A_i).
\]

Let \( x \in \bar{X} \) & \( U \ni x \) an open nbhd.

Then \( \exists \bigcap_{i \in I} U_i \) s.t. \( x \in \bigcap_{i \in I} U_i \leq U \) and almost all \( U_i = \bar{X}_i \). (Sets like this are exactly \( \bigcap_{i \in I} \pi_i^{-1}(U_i) \), finite so form a basis of the product topology).

Since the \( \bar{X}_i \) are regular, \( \exists \pi_i, v_i \subseteq \bar{V}_i \subseteq U_i \) then form \( \bigcap_{i \in I} V_i \ni x \).
(Homework 3) in any product topology
\[ \bigcap_{i \in I} \overline{V_i} = \overline{\bigcap_{i \in I} V_i} \]
therefore \[ x \in \bigcap_{i \in I} V_i \leq \bigcap_{i \in I} U_i \leq \bigcap_{i \in I} \overline{U_i} \leq \overline{\bigcap_{i \in I} U_i} \]
open closure D
Warning 4.22

One can find normal spaces $A$, $B$ s.t. $A \times B$ is not normal.
Homework ideas

S & S: 65 (not 2nd at all)

- finish topology & acabrief
- only write better argumnet
- need Besire cat