3. **Def a 3.1** A category $\mathcal{C}$ is a collection of objects $\text{Ob}(\mathcal{C})$ and a collection of morphisms or arrows $\text{Mor}(\mathcal{C})$, such that:

- For every pair of objects $a, b$ there is a set $\text{Mor}(a, b)$ of morphisms, and every morphism $f$ belongs to a unique set $\text{Mor}(a, b)$.
  
  - If $f \in \text{Mor}(a, b)$, then $a$ is the source or domain of $f$ and $b$ is the target or codomain of $f$.

*there is an associative comp. of morphisms with compatible target & source*
\[
\begin{align*}
    f &: a \to b \\
    g &: b \to c \\
    a &\xrightarrow{\text{comp}} c \\
    g &\circ f \\
\end{align*}
\]

\[
\text{Mor}_C(a, b) \times \text{Mor}_C(b, c) \xrightarrow{\text{comp}} \text{Mor}_C(a, c)
\]

**For each** \( a \in \text{Ob}(C) \), **there** is an **identity morphism** \( \text{id}_a \in \text{Mor}_C(a, a) \) s.t.

\[
f \circ \text{id}_a = f \quad \text{and} \quad \text{id}_a \circ g = g \quad \text{if}
\]
\[
f \in \text{Mor}_C(a, b) \quad \text{and} \quad g \in \text{Mor}_C(b, a)
\]

**Example 3.2**: The category of
Sets (objects are sets, morphisms are functions)

Example 3.3: The category of top spaces, $\text{Top}$. Objects are top spaces, morphisms are continuous functions.

Example 3.4: The category of groups (homomorphisms)

Example 3.5: $\text{Id}_\mathbb{D}$

Defn 3.6: A subcategory of $\mathcal{C}$ is a subcollection of objects $\text{Ob}(\mathcal{D})$. 
in \( \text{Ob} \, (C) \) & morphisms

s.t. \( \text{Mor}_C \, (a, b) \subseteq \text{Mor}_E \, (a, b) \)

If \( \text{Mor}_C \, (a, b) = \text{Mor}_E \, (a, b) \) for all \( a, b \) in \( \text{Ob} \, (D) \), we say \( D \) is a \underline{full subcategory}.

\underline{Example 3.7} Say that a topological space \((M, \tau)\)

is \underline{metrizable} if there exists a metric \( d \) on \( M \) s.t.

\( \tau \) is the associated topology. Then there is a

subcategory of \underline{metrizable spaces}, \( D \subseteq \text{Top} \)

where we take as objects only metrizable spaces &

as morphisms \( \text{Mor}_D \, (a, b) = \text{Mor}_\text{Top} \, (a, b) \).

\underline{Def 3.8} An \underline{isomorphism}
in \( C \) is a morphism
\[
\begin{array}{ccc}
a & \xrightarrow{f} & b \\
\end{array}
\]
s.t. \( f \circ f^{-1} \in \text{End}(b,a) \)
s.t. \( f \circ f^{-1} = \text{id}_b \)
\( f \circ f = \text{id}_a \).

Isomorphisms in \( \text{Set} \) are called \textit{bijective} (check).

Isomorphisms in \( \text{Top} \) are called \textit{homeomorphisms}.

\textbf{Observation 3.9} : A continuous function
\( f : X \to Y \) is a homeomorphism if and only
if $f^{-1}$ exists as a map of sets (a function) and also $f^{-1}$ is continuous.

\textbf{Example 3.10:} \\
\[ \mathbb{R}, \text{disc} \xrightarrow{id} \mathbb{R}, \{0, 1\} \]
is a continuous bijection with discontinuous inverse.

\textbf{Products} \\
\textbf{Def 3.11:} Let \( \{A_i : i \in I\} \) be a family of objects in a category \( \mathcal{C} \). A \textit{(categorical) product} of \( \{A_i : i \in I\} \) in \( \mathcal{C} \)
is a set \( \{ \cdot \} \) an object \( P \) equipped with maps \( \pi_i : P \to A_i \) such that given any other object \( B \) of maps \( f_i : B \to A_i \), there exists a unique map \( s : B \to P \) such that for all \( i \), one has \( f_i = \pi_i \circ s \), i.e.

\[
\begin{array}{ccc}
B & \xrightarrow{s} & P \\
\downarrow{f_i} & & \downarrow{\pi_i} \\
A_i & &
\end{array}
\]

The maps \( \pi_i \) are called projection maps.
Prop 3.12 Suppose $P$ & $P'$ are products of $\{A_i\}_{i \in I}$ with $\pi_i, \pi'_i$ then 3! isomorphism $s : P \rightarrow P'$ s.t.

\[
\begin{array}{cccc}
P & \rightarrow & P' \\
\pi_i & \downarrow & \pi'_i \\
A_i & \rightarrow & \pi'_i
\end{array}
\]

Proof

\[
\begin{array}{cccc}
P & \rightarrow & P' \\
\pi_i & \downarrow & \pi'_i \\
A_i & \rightarrow & \pi'_i
\end{array}
\]

$s$ exists & similarly

$s' : P' \rightarrow P$ exists.
Remark 3.12: It is not necessary that products exist in the category.

Consider the category of finite abelian groups (morphisms are homomorphisms), then
\[ \prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z} \]

cannot be constructed in this category (of course, it can be constructed in another, "larger" category).

If a product exists, then by 3.11 it's unique up to unique isomorphism.
One often speaks of the product — in a small abuse of notation.

Prop 3.13: Suppose \( \{A_i : i \in I\} \) is a family of topological spaces. Then a categorical product \( P = \prod_{i \in I} A_i \) exists. We can construct \( P \) as the set \( \prod_{i \in I} A_i \) where the elements are \( (a_i)_{i \in I}, a_i \in A_i \) (the usual set-theoretic cartesian product).

The projection maps are \( \pi_j : P \to A_j \)
\[
\pi_j ((a_i)_{i \in I}) = a_j.
\]

known as the product top.,

The topology on \( P, \pi \) is generated by all sets \( \left( \pi_i^{-1} (U) \right) \) as \( U \) ranges over open sets of \( A_i \), and \( i \) ranges over \( I \).

Proof: First, it is clear that \( P \) exists and the topology is defined. The topology is constructed in order to make the \( \pi_i \) continuous.
Now we show that this has the universal property. Suppose $B$ is a top space, $f: B \to \bigcup_{i \in I} A_i$ are continuous functions. Then define $s: B \to \mathcal{P}$ as a function of sets by $s(b) = (f_i(b)), i \in I$.

1) $\pi_i \circ s = f_i$ since $(\pi_i \circ s)(b) = f_i(b)$.

Now check $s$ is continuous: it's enough to check $s^{-1}(U)$ is open when $U$ lies in some subbasis for the topology $\mathcal{P}$ of $\bigcup_{i \in I} A_i$ but here's a subbasis: sets $\pi_i^{-1}(U)$ for $U$ open in $A_i$.

Then $s^{-1}(\pi_i^{-1}(U)) = f_i^{-1}(U)$ is open (since $f_i$ is c'tly). $\square$

**Digression 3.14** This is an example of two related things — an initial topology, and a categorical limit. Categorical limits will be discussed later.

**Definition 3.15** Let $\mathbf{X}$ be a set, let $\{A_i \mid i \in I\}$ be a family of topological...
spaces \( f_i : \overline{\mathbb{X}} \to A_i \) be a collection of sets. The initial topology on \( \overline{\mathbb{X}} \) w.r.t. \( f_i \) is the coarsest topology on \( \overline{\mathbb{X}} \) (fewest open sets) making the \( f_i \) continuous. Equivalently, it is the topology generated by \( \{ f_i^{-1}(U) \}_{i \in I} \) \( U \) open in \( A_i \).

**Prop 3.16** In the setup of 3.15, suppose we have a top. sp. \( Y \), continuous functions
\[ g_i : Y \to A_i \] and a function \( s : Y \to \overline{\mathbb{X}} \) such that \( f_i \circ s = g_i : A_i \) (i.e. \( Y \xrightarrow{s} \overline{\mathbb{X}} \xrightarrow{f_i} A_i \)).

Then if we give \( \overline{\mathbb{X}} \) the initial topology, \( s \) is continuous.

**Proof:** Here is a subbasis for the initial topology
\[ \bigcup \{ f_i^{-1}(U) : U \text{ open in } A_i \} \]
For any such set $s^{-1}(f^{-1}(U)) = g^{-1}(U)$ is open, so the result follows from ch 2. □

**Note 3.12**  If $A \subset X$ is a subset inclusion & $X$ is a topological space, then the initial topology on $A$ w.r.t. $i$ is the topology (generated by)

$$\mathcal{S} = \{ i^{-1}(U) \} \quad \text{where} \quad U \text{ open in } X \quad \text{and} \quad U \cap A $$

In this case, this $\mathcal{S}$ is already a topology, the subspace topology from ch 2. Moreover, when we take an initial top. w.r.t. one map, we as here, we say this is the topology induced by $i$. 
Prop 3.18 \[ \text{If } f : X \to Y \text{ is a c'k function and } A \subseteq Y \text{ is a subspace such that } \text{Im} f \subseteq A, \text{ then the function } \bar{f} : X \to A \text{ given by } \bar{f}(x) = f(x) \text{ is also c'k.} \]

Proof: \[ 3.16 \& 3.18, \] or can be done directly. \[ \Box \]

Prop 3.19 \[ \text{If } \{ A_i : i \in I \} \text{ are a family of top spaces and } \{ f_i : X \to A_i : i \in I \} \text{ are functions, and if } \{ U_i : i \in I \} \text{ are subbasises for the topologies on } A_i : i \in I, \text{ then the set of sets } \]
\[ F = \{ \{ f_i^{-1}(U_i) \} : U_i \in S_i, i \in I \} \]

is a subbasis for the initial topology on X.

Proof: Consider the topology \[ \tau \text{ on } X \text{ generated by } F. \text{ With this topology, all the }
functions $f_i : \mathcal{X} \rightarrow A_i$ are continuous
since $f_i^{-1}(S_i^j)$ are all open in $\tau$ where
$S_i^j$ is a subbasis for the topology on $A_i$ (see Sec. 2).
$\tau$ is at least as fine as the initial top.

On the other hand, the sets of $I$ are all open
in the initial topology, the coarsest topology
making the $f_i$ continuous. Hence the sets of
$\tau$ must be open in the initial topology,
( the initial topology is at least as fine as $\tau$).
$\square$

==

Prop 3.20  Let $(\mathcal{X}_1, d_1) \ldots (\mathcal{X}_n, d_n)$
be a finite family of metric spaces &
define

$$d^\infty : \left( \prod_{i=1}^n \mathcal{X}_i \right) \times \left( \prod_{i=1}^n \mathcal{X}_i \right) \rightarrow [0, \infty)$$

by

$$d^\infty((x_1, \ldots, x_n), (y_1, \ldots, y_n)) =$$

$$\max \{d_i(x_i, y_i), \ldots, d_n(x_n, y_n)\}$$
Then $d^\infty$ is a metric on $P$, & the metric topology is the product topology.

Proof:

The metric properties of $d^\infty$ are easy to verify for instance the triangle inequality says

$$d^\infty (\vec{x}', \vec{y}') = \max \{ d_i (x_i, y_i), \ldots, d_n (x_n, y_n) \}$$

$$\leq \max \{ d_i (x_i, z_i), \ldots, d_n (x_n, z_n) \} + \max \{ d_i (z_i, y_i), \ldots, d_n (z_n, y_n) \}$$

$$\leq \max \{ d_i (x_i, z_i), \ldots, d_n (x_n, z_n) \} + \max \{ d_i (z_i, \vec{y}'), \ldots, d_n (z_n, \vec{y}') \} = d^\infty (\vec{x}, \vec{z}) + d^\infty (\vec{z}, \vec{y}')$$

What about the metric topology for this $d^\infty$ metric?

1. Projection maps $\pi_i : P \rightarrow \overline{X}_i$ are continuous. For every $(x_1, \ldots, x_n) \in P$, for every $\varepsilon > 0$ s.t.

$$\delta = \varepsilon$$
\[ d_{\infty} \left( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \right) < \varepsilon \]

\[ = \quad d_1(x_1, y_1) < \varepsilon = \delta. \]

Hence the metric topology is at least as fine as the product topology.

2. Let \( \varepsilon > 0 \), and \( (x_1, \ldots, x_n) \in \mathcal{P} \).
Consider

\[ f_1^{-1}(B(x_1, \varepsilon)) \cap \cdots \cap f_n^{-1}(B(x_n, \varepsilon)) \]

\[ = \quad B((x_1, \ldots, x_n), \varepsilon) \]

this is open in the product topology on \( \mathcal{P} \).

so it follows that every open set in the metric topology (a union of balls) is open in the product topology.

Consequently the product and metric topologies are at least as fine as each other, so they agree. \( \square \)
Prop 3.21. Let $p \in [1, \infty)$, let $(X_i, d_i), \ldots, (X_n, d_n)$ be a family of metric spaces. Define

$$d^p : (\prod_{i=1}^n X_i) \times (\prod_{i=1}^n X_i) \to [0, \infty)$$

by

$$d^p ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \left( \sum_{i=1}^n d_i(x_i, y_i)^p \right)^{1/p}.$$ 

Then $d^p$ is a metric and $d^p, d^\infty$ (from Prop 3.17) generate the same topology (the product topology).

Proof: See supplemental notes.

Any of these may be called a product metric.

Obs 3.22. (See supplemental notes.) For any $p \in [1, \infty)$, define $(\mathbb{R}^n, d_p)$ and

$$d_p((x, y)) = \left( \sum |x_i - y_i|^p \right)^{1/p}.$$
then \( d_p \) is a metric in \( \mathbb{R}^n \), all \( d_p \)'s generate the same topology on \( \mathbb{R}^n \) & this is also the product top of the topology generated by \( d_\infty = \max \{ |x_i - y_i| \} \).

Special case \( p=2 \).

---

**Example 3.28** Give \( \mathbb{R} \) the usual topology. Let \( n \geq 1 \), then the product topology on \( \mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} \) is the \( d_2 \) metric topology

\[
d_2 \left( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \right) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} = d^2 \quad \text{(in notation of 3.20)}.
\]
Example 3.24

Let $S' = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$
given the subspace topology.

Aside: if you know from real analysis e.g. that $h(x, y) = x^2 + y^2$ is a continuous function, then $S' = h^{-1}(\mathbb{S}^1)$ is closed in $\mathbb{R}^2$.

(we know $\mathbb{S}^1 \subseteq \mathbb{R}$ is closed in several different ways).
Now define a function \( f : \mathbb{R} \to S^1 \) by 
\[
 f(x) = (\cos x, \sin x).
\]
Is \( f \) continuous?

By 8.18, \( f : \mathbb{R} \to S^1 \) is continuous if \( f' : \mathbb{R} \to S^1 \to \mathbb{R}^2 \) is continuous.

To see if \( f' : \mathbb{R} \to \mathbb{R}^2 \) \( f'(x) = (\cos x, \sin x) \) is continuous, we use 8.22 to say the topology on \( \mathbb{R}^2 \) (the metric topology) agrees with the product topology on \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \).

Consider

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{f} & \mathbb{R}^2 \\
\downarrow{\pi_2} & & \downarrow{\pi_1} \\
\mathbb{R} & \xrightarrow{\text{proj}} & \mathbb{R}
\end{array}
\]

\( f' \) is continuous \( \Leftrightarrow \) \( \pi_2 \circ f' \) are cts (3.16 & 3.17)

so \( f' \) is continuous \( \Leftrightarrow f \) is continuous, but \( \sin x \) and \( \cos x \) are continuous functions, but they are differentiable.

so \( f : \mathbb{R} \to S^1 \) is continuous.
Example 3.25. If \( \{ A_i : i \in I \} \) and \( \{ B_i : i \in I \} \) are two families of topological spaces and \( f_i : A_i \to B_i \) are families of functions, then for each \( i \) there is a unique function

\[
\prod_i A_i \xrightarrow{\pi_i} A_i \xrightarrow{f_i} B_i
\]

\( i \in I \)

there is a unique function \( s : \prod_i A_i \to \prod_i B_i \) such that

\[
\prod_i A_i \xrightarrow{\pi_i} A_i \xrightarrow{f_i} B_i
\]

for all \( i \in I \). \( s ((a_i)_{i \in I}) = (f_i(a_i))_{i \in I} \). If the \( f_i \) are all continuous, \( f_i \) and \( \pi_i : \prod_i A_i \to B_i \) are all continuous \( \Rightarrow s \) is continuous.
**Quotient Spaces**

**Def 3.26** Let \( \{ \Lambda; \} \}_{i \in \mathbb{I}} \) be a family of topological spaces \( \Lambda \) \( \rightarrow \) \( \overline{\Lambda} \) a set; let \( \{ A_i \xrightarrow{f_i} \overline{X} \}_{i \in \mathbb{I}} \) be a family of functions. The final (or colimit) topology on \( \overline{X} \) is the topology consisting of those sets \( U \subseteq \overline{X} \) s.t. \( (f_i^{-1}(U) \text{ open}, \forall i \in \mathbb{I}) \) — the finest topology for which the \( f_i \) are cts.
Prop 3.27: The final topology is a topology; and it makes the functions \( f_i \) continuous.

Proof: Easy; done out in lecture.

Prop 3.28: Given the setup of 3.26, suppose \( Y \) is a top. space \( \mathcal{F} \)

\( \{g_i: \Lambda_i \to Y\} \) is a collection of functions \( s \)

\[
\begin{bmatrix}
    f_i & A_i & g_i \\
    \alpha & c_i & y \\
    \beta & \gamma & \delta
\end{bmatrix}
\]

then \( s \) is continuous.
Proof: This is the purpose of the final topology. Suppose \( U \) is open in \( Y \), then \( s^{-1}(U) \) satisfies
\[
\forall i : (s_i^{-1}(U)) = s_i^{-1}(U) \text{ open } \forall i ;
\]
\[
\Rightarrow s^{-1}(U) \text{ is open. } \quad \square
\]

The final topology is the unique topology making 3.27, 3.28 hold.
Def 3.29] Suppose $f: X \to Y$ is a continuous surjective function of top. sp. we say $f$ is a quotient map if $Y$ has the final topology for $f$. When the topology on $Y$ is the final topology w.r.t one map, we say the topology is the coinduced topology.

Example 3.30] (3.24 continued)
Consider $\mathbb{R}^2$, $S' = \{(x, y) : x^2 + y^2 < 1\}$ and $f: S' \rightarrow (\cos \theta, \sin \theta)$. This is a $C^k$ by 3.24. It is also surjective (use arc$\cos(x)$ or arc$\sin(y)$).

We claim $S'$ has the topology coinduced by $f$, so that $f$ is actually a quotient map. (Topologically)

The coinduced topology on $S'$ is the topology where $U \subseteq S'$ is open if and only if $f^{-1}(U) \subseteq \mathbb{R}^2$ is open. We need to show this is the 'usual' topology on $S'$, the subspace topology for $S' \subseteq \mathbb{R}^2$.

It's enough to show $f^{-1}(U)$ open in $\mathbb{R}^2 \Rightarrow U$ open in usual topology on $S'$. The other direction is handled by knowing $f$ is continuous.

But by H/2.2, we know somewhere, $f$ is an open map (quotient maps are often open) so for any $U \subseteq S'$, if $f^{-1}(U)$ is open in $\mathbb{R}^2$
then \( f \circ (f^{-1}(U)) = U \) is open in \( S^1 \). \( \square \)

Is the function \( f : I \rightarrow S^1 \), \( f(x) = (\cos(2\pi x), \sin(2\pi x)) \) continuous?

Consider \( g : I \rightarrow \mathbb{R}^2 \), \( g(x) = (\cos(2\pi x), \sin(2\pi x)) \)

I claim this is continuous.
usual metric topology.

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the product topology

(Prop 3.20), so to show $\tilde{f} : I \rightarrow \mathbb{R}^2$ is $C^1$, it's enough to show $\pi_i \circ \tilde{f}$ is $C^1$ for $i = 1, 2$. (product topology is initial $\mathcal{J}$, $\mathcal{B}$)

$\pi_i \circ \tilde{f} = \cos 2\pi x \quad \pi_i \circ \tilde{f} = \sin 2\pi x$

continuous by $\varepsilon-\delta$ & real analysis (these functions are smooth).

$$(g^{-1})^{-1}(u) = g(u)$$

$$g\left((x-\varepsilon), (x+\varepsilon)\right)$$

$$2\pi i \varepsilon + 2\pi i \varepsilon$$
\[ g(\{x\} \cup \{0, \varepsilon\} \cup \{-\varepsilon, 1\}) \]

Note that \( g(\alpha) \) always lies in a single open semicircle.

Suppose \( g(\alpha) \) then we neary define this as \( g'' \).
\[ S^1 \cap \Pi^{-1}_x ( \cos 2\pi (t + \varepsilon), \cos 2\pi (t - \varepsilon) ) \]

\[ \cap \Pi^{-1}_y ((0, \infty)) \]

so it is open on \( S^1 \). The other cases are similar.

\[ I/_{0\sim1} \xrightarrow{g} S^1 \]

is a homeomorphism.
Defn 3.25) Given a topological space $(X, \tau)$ and a subset $A$, define $\overline{X/A}$, the quotient of $X$ by $A$, as

$$(X - A) \cup \{ \ast \times \frac{1}{2} \}$$

equipped with a map

$$\overline{X} \xrightarrow{\pi} \overline{X}/A$$

$$\pi(x) = x \quad \text{if} \quad x \not\in A$$

$$\pi(x) = \ast \quad \text{if} \quad x \in A.$$
(This is the quotient topology provided \( A \) is not empty).

Then \( I/\{0,1\} \cong S^1 \).

---

**Def. 3.26** A map \( f : Y \rightarrow X \) is said to be open (resp. closed) if (resp. closed) \( f(U) \) is open, in \( X \) for all open (resp. closed) \( U \) in \( Y \).
Def 2.16. Let $C$ be a category (e.g. $\text{Top}$). Let $\{A_i\}_{i \in I}$ be a set of objects of $\{f_j\}_{j \in J}$ a set of morphisms having sources & targets in $\{A_i\}_{i \in I}$. We will say $\{(A_i, f_j)\}$ is a diagram.

E.g.

\[
\begin{array}{cc}
A_0 & \xrightarrow{f_1} & A_1 \\
\downarrow{f_2} & & \downarrow{f_2} \\
A_1 & \xrightarrow{f_1} & A_2
\end{array}
\]
We say an object $L$ is a family of morphisms $\{\pi_i : L \to A_i : i \in I\}$ is a limit for the diagram, if:

1. For every composable chain of arrows $A_{i_1} \xrightarrow{f_{i_1}} A_{i_2} \to \cdots \to A_{i_n}$ in the diagram

   $\begin{array}{ccc}
   & L & \\
   \pi_i & \searrow & \pi_i \\
   A_{i_1} & \to & A_{i_n}
   \end{array}$

   commutes

2. If $B, \{f_i : B \to A_i : i \in I\}$ is any other object and family of maps satisfying property 1, then there exists a unique morphism $s : B \to L$ such that $\pi_i \circ s = f_i$ for all $i \in I$. 