Definition 2.1] Let \( X \) be a set and \( \tau \) a set of subsets of \( X \). We say \( (X, \tau) \) is a topological space if \( \tau \) satisfies:

P1. If \( \{ U_i : i \in I \} \subseteq \tau \), then
\[
\bigcup_{i \in I} U_i \in \tau \quad (\text{each } U_i \text{ open})
\]

P2. If \( \{ U_i, \ldots, U_n \} \subseteq \tau \), then
\[
\bigcap_{i=1}^{n} U_i \in \tau
\]

P3. \( \{ \emptyset, X \} \subseteq \tau \).

The sets \( \tau \) are open sets of the topology. The set \( \tau \) is called "a topology."
Note 2.2: If \((X, d)\) is a metric space then the set of open sets of \(X, \tau\) forms a topology — so \((X, \tau)\) is a topological space. This is called the metric topology.

Two different metrics may give rise to the same topology — we'll see examples later.

Not all topologies are metric; but the most important ones often are.

Example 2.3: Let \(X\) be a set and \(\tau = \{\emptyset, X\}\). Then \((X, \tau)\) is
a topological space. If \( X \) has at least 2 elements, then this is not a metric topology.

**Example 2.4:** Let \( X \) be an infinite set and let \( \tau \) consist of those sets \( U \subseteq X \) s.t. \( |X \setminus U| \) is finite or \( U = X \). Then \( \tau \) is a topology, the **cofinite topology.** It is not metric.

**Def 2.5:** If \((X, \tau)\) is a topological space and \( C \) is a subset of \( X \), s.t. \( X \setminus C \in \tau \) is open, then we say \( C \) is **closed.**
Prop 2.6: Let $\mathcal{K}$ denote the set of closed sets of $(\mathbb{R}, \tau)$.

$P_4$: If $\{C_i\}_{i \in I} \subseteq \mathcal{K}$ then

$$\bigcup_{i \in I} C_i \in \mathcal{K}$$

(arbitrary union of closed is closed)

$P_5$: If $\{C_1, \ldots, C_n\} \subseteq \mathcal{K}$ then

$$\bigcup_{i=1}^n C_i \in \mathcal{K}$$

$P_6$: $\{\emptyset, \mathbb{R}\} \subseteq \mathcal{K}$.

Proof: Exercise.
Def 2.8: Let $(\mathcal{X}, \tau_1), (\mathcal{Y}, \tau_2)$ be top spaces. We say a function $f : \mathcal{X} \to \mathcal{Y}$ is continuous if $f^{-1}(U)$ is open in $\mathcal{X}$ for all open $U \subseteq \mathcal{Y}$.

Note 2.8: If $[\mathcal{X}, \tau_1], (\mathcal{Y}, \tau_2)$ are metric topologies, then this definition agrees with the metric definition of continuity.

Prop 2.9: $(\mathcal{X}, \tau_1), (\mathcal{Y}, \tau_2), (\mathcal{Z}, \tau_3)$

$f : \mathcal{X} \to \mathcal{Y}, g : \mathcal{Y} \to \mathcal{Z}$

both $c, f$
then \[ g \circ f : X \to \mathbb{R} \text{ is continuous.} \]

**Proof**: Exercise.

From here on, we will write \((\overline{X}, \overline{\mathcal{C}})\) or \(\overline{X}\). etc.

**Prop 2.10**: \( f : \overline{X} \to Y \) is continuous if and only if \( f^{-1}(C) \) is closed for all closed \( C \subseteq Y \).

**Examples 2.11**: Let \( X \) be a set with the discrete metric. Then every subset is open (since \( \{ x \} = B(x; \frac{1}{2}) \), every subset contains a ball around every point).

Let \( f : \overline{X} \to Y \) be a
function from $\bar{X}$ to a top. sp. $Y$
Then $f$ is c'13.

( the discrete topology)

Example 2.12: Let $Y$ be equipped with the indiscrete topology, let $\bar{X}$ be a topological space & $f: \bar{X} \to Y$
a function. Then $f^{-1}(\emptyset) = \emptyset$
$\forall$ $y \in Y$ & $f^{-1}(\{y\}) = \{x \in \bar{X} | f(x) = y\}$ open

$\Rightarrow$ $f$ is c'13

Example 2.13: $id_{\bar{X}}$ is continuous.

More concepts for topologies

Let $(\bar{X}, \tau)$ denote a topological space throughout.
**Definition 2.15**: If \( A \subseteq \bar{X} \), we may endow \( A \) with the *subspace topology* \( \tau|_A = \{ U \subseteq A \mid U \in \tau, \forall A \in \tau \} \).

**Observation 2.14**: The inclusion map \( i : A \rightarrow \bar{X} \) is continuous, \( \tau|_A \) is the coarsest topology for which \( i \) is continuous.
Definition 2.16: Suppose $A \subseteq \mathbb{X}$.
Define the closure of $A$, $\overline{A}$, as the intersection $\bigcap C_i$ of all closed sets containing $A$.

Prop 2.16: Closure has the following prop.
- $\overline{A}$ is closed, $A \subseteq \overline{A}$
- If $A$ is closed, $\overline{A} = A$.
- In particular $\overline{\overline{A}} = \overline{A}$.
- If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.

Proof: $\overline{A}$ is the intersection of a family of closed sets, and is therefore closed.
- $A \subseteq \overline{A}$ is obvious from the def.
- If $A$ is closed then $\overline{A} = A$. 
\[ \Rightarrow A = A \]

- If \( A \subseteq B \) then every closed set containing \( B \) contains \( A \)

\[ \bigcap C \subseteq \bigcap C \]
contains \( A \)
contains \( B \) & \( A \)
intersection of a subfamily \( D \)

The dual concept of closure is that of interior.

**Def 2.17** \( \text{Int}(A) \) or \( A^o \) is the union of all open sets contained in \( A \).

\[ 2.18 \text{ Int} \left( \text{Int}(A) \right) = \text{Int}(A) \]
\[ \text{Int}(A) \text{ is open} / \text{Int of open } A \text{ is } A \text{ again.} \]
\[ \bar{X} - \text{Int}(\bar{X} - A) = \overline{A} \]
\[ \bar{X} - \overline{X - A} = \text{Int}(A) \]

2.19. **Defn | \( \partial A \), the boundary of \( A \), is \( \overline{A} - \text{Int}(A) \)**

Any space \( A \subseteq \bar{X} \) breaks \( \bar{X} \) into three parts

\[ \begin{array}{c}
\text{Int} A \\
\hline
\partial A \\
\text{Int} A
\end{array} \]

\[ \bar{X} - \overline{A} = \text{Int}(\bar{X} - A) \]

---

Arg 2.20. If \( A \) is a set \( \subseteq \bar{X} \), then
\[ a \in \text{Int}(A) \iff \text{an open set } a \in U \subseteq A. \]

\[ a \in \overline{A} \iff \text{any open sets } U \ni a, \ U \cap A \neq \emptyset \]
\[ \implies a \in \partial A \iff \text{any open sets } U \ni a, \ U \cap A \neq \emptyset \land U \cap (\overline{X} - A) = \emptyset. \]

Prove \text{ Interior, then \gamma.}

2.21: Notation: an \textit{(open)} neighborhood of \( x \in \overline{X} \) is an open set \( U \ni a. \)

*(To save people's blood means a set containing an open noof.)*
Base of subbase, generally topology

Def: 2.22 A basis for a topology $\mathfrak{X}$ is a set $\mathcal{B}$ of open sets of $(\mathfrak{X}, \mathcal{O})$ s.t.

$\forall$ open $U$, $\forall \ x \in U$, $\exists \ B \in \mathcal{B}$ s.t. $x \in B \subseteq U$.

Balls $B(x, \varepsilon)$ form a basis for a metric topology.

$U = \bigcup_{x \in U} B(x)$
$x \in U$

**Def 2.25**] We say $\mathcal{B} \subseteq \mathcal{T}$ is a subbasis for a topology $\mathcal{T}$ if for every $U \in \mathcal{T} \& x \in U$

$\exists \{S_1, S_2, \ldots, S_n\} \in \mathcal{B}^* \& x \in (S_1 \cap S_2 \cap \ldots \cap S_n) \subseteq U$.

i.e., finite intersections of sets in $\mathcal{B}$ form a basis.
Note 2.26) If \( U \) is open in \( \mathbb{R} \) and \( S \) is a subbasis, then for any \( x \in \mathbb{R} \) are only \( S(\{x\})_1, \ldots, S(\{x\})_n \)

s.t. \( x \in S(\{x\})_1 \land \ldots \land S(\{x\})_n \subseteq U \)

Then \( U = \bigcup_{x \in U} \left( \bigcap_{i=1}^{n} \{x\}_i \right) \)

so \( U \) is a union of finite intersections.

Conversely, any such \( U \) must be open. Therefore, the subbasis determines the topology.

Every basis is a subbasis.
Prop 2.26] Suppose \( D \) is a special basis for a topology on \( X \) and \( f : Y \rightarrow X \) is a function from a top sp. \( Y \) to \( X \). Then:

\( f^{-1}(D) \) is open \( \forall S \subseteq D \)

if \( f \) is c'fs.

Proof: We make use of the following observables:

\[ f^{-1}(\bigcup W_i) = \bigcup f^{-1}(W_i) \]
\[ f^{-1} \left( \bigcup W_i \right) = \bigcup f^{-1}(W_i) \]

for families of subsets of \( Y \).

We can write open sets \( U \subseteq \mathbb{R} \) as

\[ U = \bigcup_{s \in S} \cap \text{finite} \]

\[ f^{-1}(U) = \bigcup_{s \in S} \cap f^{-1}(s) \]

so \( f^{-1}(U) \) is open.
Theorem 2.27: If \( \{ T_i \} \) is a family of topologies on \( X \), then \( \bigcup T_i \) is a topology. (easy proof)

This implies that given any set \( S \subseteq \mathcal{P}(X) \), there is a coarsest topology on \( X \) such that the sets in \( S \) are all among the open sets.

Note:
Prop 2.28: If \( S \subseteq \mathcal{X} \), then
define \( \tau(S) \) to be the set of all
unions of finite intersections of sets in \( S \).
Including \( \emptyset, \mathcal{X} \). Then \( \tau(S) \) is
the coarsest topology on \( \mathcal{X} \) containing \( S 
\) (in particular, it is a topology), and
Moreover, \( S \) is a subbasis for \( \tau(S) \).
Proof: set $\sigma_1 \cap (\bigcup (\bigcap S_{j_i})) = U$

the only thing to check is finite intersections of these are again of this form, but identify $\bigcap$ as the topology. Plainly these sets must be open in any topology. The sets $S$ are open. Finally, given any $x \in U$ where $U$ is open in $\tau_1(\sigma_1)$, $x \in U(\bigcup S_{j_i}) = U$.

$= \exists$ some $S_1, \ldots, S_n$ s.t.
$x \in (S_1 \cup \ldots \cup S_n) \leq U$.

Example 2.29: We can now make a huge range of topological spaces by specifying a set $\mathcal{S}$ and a subbasis $\mathcal{B} \subset \mathcal{P}(\mathcal{S})$.

Then $\tau(\mathcal{B})$ is the set of all unions of finite intersections of sets in $\mathcal{B}$. For instance, define $(\mathbb{R}, \tau)$ to be the topology generated by left-half-open intervals $(a, b]$ (in fact, since $(a, b] \cap (c, d]$ is again half-open, this collec...
of sets forms a base for a topology.

Since one may write

\[(a, b) = \bigcup_{n \in \mathbb{N}} (a, b - \frac{1}{n}) ,\]

it follows that the 'open intervals' are open sets for the half-open topology \((\mathbb{R}, \mathcal{T}_e)\)

\[\Rightarrow (\mathbb{R}, 1.1) \xrightarrow{id} (\mathbb{R}, 1.1)\]

continuous by prop 2.26.

ordinary tp

basis of open intervals \((a - \varepsilon, a + \varepsilon)\)

\[\left(\mathbb{R}, 1.1\right) \xrightarrow{id} (\mathbb{R}, \mathcal{T}_e)\]