1: Metric Spaces

Definition 1: A metric space consists of a set \( \mathbb{X} \) and a function

\[ d: \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty) \]

satisfying the following axioms

1. \( d(x, y) = d(y, x) \quad \forall x, y \in \mathbb{X} \) (symmetry)

2. \( d(x, y) = 0 \quad \text{iff} \quad y = x \) (identity)

3. \( d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in \mathbb{X} \) (triangle inequality)

Example 1.2: Let \( \mathbb{X} = \mathbb{R}^n \) \((n \geq 1)\)

and let

\[ d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \]

(ordinary Euclidean distance)
Then \((\overline{X}, d)\) is a metric space. 
\((\overline{\mathbb{R}}, d_2)\) 
(\(\Delta\)-inequality is an exercise).

Example 13: Let \(\overline{X} = \mathbb{R}^n\), \(n \geq 1\), def \(p \geq 1\) and define
\[
d_p((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}
\]
(when \(p = 2\), this recovers the previous example) \(p\)-metric.

Example 14: Let \(\overline{X} = \mathbb{R}^n\)
\[
d_{\infty}(\overline{x}, \overline{y}) = \max_{i \in \{1, \ldots, n\}} \{x_i - y_i\}
\]

Example 15: Let \(\overline{X}\) be any set and def
\[
d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}
\]
Definition 1.6: Let \( y \leq x \in \mathbb{X} \) be an archic in \( \mathbb{X} \). Then
\[
( y, d_{\mathbb{Y}}(y) ) \quad \text{where} \quad d_{\mathbb{Y}}(y_1, y_2) = d(y_1, y_2)
\]
is again an archic space.

Any subset of \((\mathbb{R}^n, \mathfrak{d}_2)\) is an archic space. This is the source of most early examples.

Note: Since \((\mathbb{X}, \mathfrak{d})\) is
a metric space $(X, d)$ and $r > 0$

The (open) ball centred at $x$ of radius $r$, denoted $B(x; r)$, is defined as $\{ y \in X \mid d(x, y) < r \}$.

Ref 1.9: The closed ball $\overline{B}(x; r)$ is defined as $\{ y \in X \mid d(x, y) \leq r \}$.

Definition 1.9: A metric space $(X, d)$ is bounded if $\exists \ r > 0, x \in X
S.t. \ B(x; r) = X$.

Exercise 1.10: If $(X, d)$ is bounded,
then for any \( x \in \mathbb{R} \) and \( r > 0 \) s.t.
\[
B(x; r) = \overline{X}.
\]

**Notation:** if \( f : \overline{X} \to \mathbb{R} \) is an \( A \subseteq \mathbb{R} \), define \( f^{-1}(A) = \{ x \in \overline{X} \mid f(x) \in A \} \) — the pre-image of \( A \).
\[
f^{-1}(y) \overset{def}{=} f^{-1}(\{y\}).
\]

**Definition 1.11:** Suppose \( (\overline{X}, d_1) \), \( (\mathbb{R}, d_2) \) are metric spaces, and \( f : \overline{X} \to \mathbb{R} \) is a function. We say \( f \) is **continuous** if \( x \in \overline{X} \)
\[
\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t.}
\]
\[ \forall x', \text{ s.t. } d_1(x, x') < \delta \]
we have \[ d_2(f(x), f(x')) < \varepsilon. \]

**Definition 1.12.** If \( f : X \to Y \) is continuous at every \( x \in X \), we say \( f \) is continuous.

**Theorem 1.13.** With \( (X, d_1), (Y, d_2) \),

- \( f \), as above.

TEAE

1. \( f \) is \( C^1 \).

2. \( \forall x \in X, \forall \varepsilon > 0, \exists \delta \)
\[ f^{-1}(B(f(x), \varepsilon)) \supset B(x, r_{x, \varepsilon}). \]

**Definition 1.14:** If \((X, d)\) is metric space, and \(U \subseteq X\), we say \(U\) is open (for \(d\)) if, for all \(u \in U\), \(\exists \varepsilon > 0\) s.t. \(B(u, \varepsilon) \subseteq U\).

**Properties 1.15:** Let \((X, d)\) be a metric space; let \(\mathcal{T}\) denote the set of all open sets of \(X\).

**P1:** Suppose \(\{U_i\}_{i \in I} \subseteq \mathcal{T}\), then \(\bigcup_{i \in I} U_i\) is open.
P2: Suppose \( \{U_1, \ldots, U_n\} \subseteq \mathbb{R} \)

Then \( \bigcap_{i=1}^{n} U_i \) is open (finite int.)

P3: \( \emptyset, \mathbb{R} \) are both open.

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Proof of Properties:

P1: is easy enough. Let \( x \in \bigcup_{i \in I} U_i \)

then \( x \in U_i \) for some \( i \in I \) =>

\[ \exists \, \epsilon > 0 \text{ s.t. } B(x, \epsilon) \subseteq U_i \]

\[ \Rightarrow B(x, \epsilon) \subseteq \bigcup_{i \in I} U_i \]

P2: Suppose \( x \in \bigcap_{i=1}^{n} U_i \)

then \( x \in U_i \) \( \forall \, i \in I \)
⇒ ∃ \varepsilon_i \text{ s.t. } B(x, \varepsilon_i) \subseteq U_i

let \varepsilon_m = \min \{\varepsilon_1, \ldots, \varepsilon_n\}

Note \varepsilon_m exists because \varepsilon_1, \ldots, \varepsilon_n
is finite.

B(x, \varepsilon_m) \subseteq B(x, \varepsilon_i) \subseteq U_i

\Rightarrow B(x, \varepsilon_m) \subseteq \bigcap_{i \in \mathbb{C}} U_i

so \bigcap_{i \in \mathbb{C}} U_i \text{ is open.}

\underline{P3} is valid.

Prop 1.16 | Open balls \( B(x, r) \) are open.

Proof: let \( y \in B(x, r) \)
\( s = d(x, y) < r \)

Choose \( \varepsilon > 0 \) s.t. \( 0 < \varepsilon < r - s \)

Suppose \( d(y, z) < \varepsilon \), then
\[
d(x, z) \leq d(x, y) + d(y, z) < s + \varepsilon < r
\]
\( \Rightarrow z \in B(x, r) \implies B(y, \varepsilon) \subseteq B(x, r) \)
so \( B(x, r) \) is open. \( \Box \)

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Theorem 1.17 \( ^{\dagger} \)

Let \( f : X \rightarrow Y \) be a map of metric spaces.

\( f \) is c'ls

\( \implies \)

\( \forall \) open sets \( U \subseteq Y \),

\( f^{-1}(U) \) is open in \( X \).

Proof: \( \implies \) Assume \( f \) c'ls.
Let $U$ be open in $Y$, consider $f^{-1}(U)$. For any $x \in f^{-1}(U)$ we have $f(x) \in U \Rightarrow \exists \varepsilon > 0$ s.t. $B(f(x), \varepsilon) \subseteq U$.

Since $f$ is c'ts, by The 1.3

$\exists \delta > 0$ s.t. $B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon)) \subseteq U$.

So $U$ contains an open ball around any of its points, is open.

\[ \square \]

Suppose $f^{-1}(U)$ is open for all open $U \subseteq Y$. Let $x \in X$, $\varepsilon > 0$.

The set $B(f(x), \varepsilon)$ is open

and $f^{-1}(B(f(x), \varepsilon))$ is open by
hypothesis. So new, since
\[ x \in f^{-1}(B(f(x), \varepsilon)) \]
\[ \exists \delta > 0 \text{ st.} \]
\[ B(x, \delta) \subseteq f^{-1} B(f(x), \varepsilon) \]