Chapter 9

The van Kampen theorem

The basic problem is as follows. Suppose \( X = U \cup V \) where \( U \) and \( V \) are open sets such that \( U \cap V \), \( U \) and \( V \) are connected. Let \( x_0 \in U \cap V \) be a point. Can we determine \( \pi_1(X, x_0) \) from \( \pi_1(U, x_0) \), \( \pi_1(V, x_0) \) and \( \pi_1(U \cap V, x_0) \)?

The answer is yes, and in fact, we can do better.

First of all, recall that \( \Pi(X) \) denotes the fundamental groupoid of the space \( X \).

**Proposition 28.10.** Let \( X \) be a topological space and let \( U, V \) be open subspaces with \( X = U \cup V \). Let \( G \) be a groupoid and suppose that in the diagram below, the outer square commutes. Then there exists a unique map of groupoids indicated by \( f \) making the whole diagram commute.

\[
\begin{array}{ccc}
\Pi(U \cap V) & \xrightarrow{f} & \Pi(U) \\
\Pi(V) & \xrightarrow{h} & \Pi(X) \\
& \xrightarrow{g} & G
\end{array}
\]

**Proof.** A map of groupoids is a functor, therefore a function of objects and of morphisms. In this case, the story about objects (the points of the spaces) is easy. The hard part is about paths.

We assume we have produced the (unique) function \( f : X \to \text{ob}G \). Now we have to worry about paths. Let \( \gamma : I \to X \) be a path in \( X \), from \( a = \gamma(0) \) to \( b = \gamma(1) \). We can find some decomposition of \([0,1]\) into closed subintervals \([t_0 = 0, t_1], [t_1, t_2], \ldots, [t_{r-1}, t_r = 1]\) such that for any \( i \), the path \( \gamma|_{[t_i, t_{i+1}]} \) lies either in \( U \) or in \( V \). Let \( \gamma_i \) denote a reparametrization of \( \gamma|_{[t_i, t_{i+1}]} \). This allows us to define a candidate \( f(\gamma) \) as the composite of the images in \( G \) of \( \gamma_0, \gamma_1, \ldots, \gamma_r \). We have not shown that \( f(\gamma) \) is well defined, but let us observe that the definition of \( f(\gamma) \) is unchanged if we replace our decomposition of \( I \) by a refinement.

Observe that if a map \( f : \Pi(X) \to G \) of groupoids exists making the diagram commute, then it must be this one, since we have factored \( [\gamma] = [\gamma_0][\gamma_1] \cdots [\gamma_{r-1}] \), and what \( f \) does to \( [\gamma_0], [\gamma_1], \ldots, [\gamma_{r-1}] \) is forced on us by \( h \) and \( g \).

Now let us prove that \( f(\gamma) \) is an invariant of the homotopy type (relative to \([0,1] \)) of \( \gamma : I \to X \). That is, if we choose a possibly different path \( \gamma' : I \to X \) such that \( \gamma = |\gamma'| \) and a decomposition of \( \gamma' \), we obtain the same definition of \( f(\gamma) \).
Suppose we have two paths \( \gamma \) and \( \gamma' \), a basepoint-preserving homotopy \( H : I \times I \to X \) and two decompositions of \( I \) as above. Using the Lebesgue covering lemma, we can find a tessellation of \( I \times I \) into small rectangles \( R_{ij} \) with disjoint interiors so that for each such rectangle \( H|_{R_{ij}} \) lies either in \( U \) or in \( V \), and so that the restrictions of the tessellation to \( I \times \{0\} \) and \( I \times \{1\} \) refine the two decompositions of \( I \). Now consider \( H|_{R_{ij}} \) in each \( R_{ij} \). They all give a relation in the fundamental groupoid either of \( U \) or of \( V \): namely \( H|_{\text{bottom}} + H|_{\text{light}} = H|_{\text{left}} + H|_{\text{top}} \). Applying either \( h \) or \( g \), as required, each \( R_{ij} \) gives us a relation in the groupoid \( G \). Integrating these relations together over the whole square shows that \( f(\gamma) = f(\gamma') \).

The proof that \( f \) is really a map of groupoids is not difficult. One simply has to verify that it sends constant paths in \( X \) to identity maps in \( G \)—which is immediate—and that it preserves composition. The statement that it preserves composition follows by taking a composite \( [\gamma][\delta] \) and decomposing each into short paths lying either in \( U \) or \( V \): \( [\gamma_0] \cdots [\gamma_{r-1}] [\delta_0] \cdots [\delta_{s-1}] \). By construction \( f([\gamma][\delta]) \) is the product in \( G \) of \( f(\gamma_0) \cdots f(\delta_{s-1}) \) equals \( f(\gamma)f(\delta) \).

\[\square\]

**Corollary 28.11.** Let \( X \) and \( U, V \) be as above. Let \( A \subseteq U \cap V \) be a set of points such that each of path component of \( U, V \) and \( U \cap V \) contains at least one point of \( A \). Let \( G \) be a groupoid, and again suppose the outer diagram commutes:

\[
\begin{array}{ccc}
\Pi(U \cap V, A) & \longrightarrow & \Pi(U, A) \\
\downarrow & & \downarrow \\
\Pi(V, A) & \longrightarrow & \Pi(X, A) \\
\downarrow & & \downarrow \quad f \\
\Pi(U, A) & \longrightarrow & \Pi(X) \\
\end{array}
\]

Then there exists a unique map of groupoids making the diagram commute.

**Proof.** For each \( x \in (U \cap V) \setminus A \), choose a fixed isomorphism in \( \Pi(U \cap V, A) \) from \( x \) to some point \( a \in A \), \( \gamma : x \to a \). By means of this isomorphism, we construct a map of groupoids \( \Pi(U \cap V) \to \Pi(U \cap V, A) \). We can do the same for \( U \) and \( V \) and \( X \), and where applicable choose the same isomorphisms.

The rest of the argument follows by showing that the diagram in the corollary is a retract of the diagram in the proposition. This is a diagram chase that is best done live. We give the diagram here and advise the reader to do the chase.
Suppose the diagram with red arrows is given, then one can construct the solid cyan arrows, then using Proposition 28.10, one constructs the dashed cyan arrow. By composing, one obtains the dashed green arrow. One then verifies that this is the map \( f \) asked for in the corollary.

\[ \text{Remark 28.12.} \] The corollary determines \( \Pi(X, A) \) up to unique isomorphism. It is the groupoid one obtains generated by the maps in \( \Pi(U, A) \) and in \( \Pi(V, A) \) subject to the relations \( \Pi(U \cap V, A) \).

\[ \text{Example 28.13.} \] Cover \( S^1 \subset \mathbb{C} \) by two open sets, \( S^1 \setminus \{i\} \). Let \( A \) be the set of points \( \{\pm 1\} \). While \( S^1 \setminus \{i\} \) and \( S^1 \setminus \{-i\} \) are both contractible, \( S^1 \setminus \{i, -i\} \) is a disjoint union of two contractible sets. The fundamental groupoids are

\[
\Pi(U, A) = \{f: -1 \to 1 : f^{-1}\} \quad \Pi(V, A) = \{g: -1 \leftrightarrow 1 : g^{-1}\} \quad \Pi(U \cap V, A) = \{-1, 1\}
\]

Therefore the corollary says that the fundamental groupoid of \( S^1 \) on the points \( \{i, 1\} \) has two objects and two different morphisms \( f, g: -1 \to 1 \). In particular, if we restrict to the basepoint 1, we see that the fundamental group of \( (S^1, 1) \) is infinite cyclic generated by \( fg^{-1} \).

\[ \text{Construction 28.14.} \] Let \( G \) and \( H \) be two groups. A word in \( G \) and \( H \) is a string of elements \( s_1 \ldots s_n \), each one either in \( G \) or \( H \). They are subject to reduction, i.e., removing an identity element or replacing a pair \( g_1 g_2 \) by its product in \( G \), or similarly in \( H \). A reduced word is a word that cannot be reduced further.

The free product \( H \ast G \) is the group of reduced words, with concatenation-followed-by-reduction as an operation.

\[ \text{Definition 28.15.} \] Let \( G, H \) be two groups and \( i_1 : J \to G, i_2 : J \to H \) be inclusions of a third group. The notation \( G \ast_J H \) denotes the amalgamated product of \( G \) and \( H \) over \( J \). This is the quotient of \( G \ast H \) by the subgroup generated by elements \( i_1(j) i_2(j)^{-1} \) as \( j \) ranges over the elements in \( J \). This is the universal group such that a unique arrow exists in commutative diagrams:

\[
\begin{array}{ccc}
J & \to & G \\
\downarrow & & \downarrow \\
H & \to & G \ast_J H \\
\downarrow & & \downarrow \\
& & \Gamma \\
\end{array}
\]

\[ \text{Corollary 28.16 (The van Kampen theorem).} \] Let \( X \) be a topological space with basepoint \( x_0 \), and \( U, V \) path connected open subsets that cover \( X \) and such that \( U \cap V \) is path connected and contains \( x_0 \). Then \( \pi_1(X, x_0) = \pi_1(U, x_0) \ast_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \).

\[ \text{Example 28.17.} \] Let \( X \) be a space that can be covered by two simply connected subsets with path connected intersection. Then \( X \) is simply connected. As a special case, \( \pi_1(S^n, s_0) = \{e_{s_0}\} \) for all \( n \geq 2 \).