Chapter 8

Covering spaces and the fundamental groupoid

24 The fundamental groupoid

Construction 24.1. Suppose $\gamma, \delta : I \to X$ are two paths, and suppose $\gamma(1) = \delta(0)$. We define a composite path $\gamma \cdot \delta : I \to X$ by
\[
\gamma \cdot \delta(t) = \begin{cases} 
\gamma(2t) & \text{if } t \leq 1/2 \\
\delta(2t - 1) & \text{if } t \geq 1/2
\end{cases}.
\]

Notation 24.2. Let us say that two paths $\gamma, \gamma' : I \to X$ are equivalent and write $\gamma \simeq \gamma'$ if $\gamma \simeq \gamma'$ relative to $\{0, 1\}$. In particular, $\gamma$ and $\gamma'$ have the same endpoints. We may write $[\gamma]$ for the equivalence class of $\gamma$.

What follows is some technical lemmas about path composition.

Proposition 24.3. If $\gamma \simeq \gamma'$ and $\delta \simeq \delta'$ and if $\gamma \cdot \delta$ is defined, then $\gamma \cdot \delta \simeq \gamma' \cdot \delta'$.

Remark 24.4. That is, the composition of paths descends to equivalence classes. After we have proved this result, we can define $[\gamma][\delta]$ by choosing representatives.

Proof. Let $H$ be a homotopy from $\gamma$ to $\gamma'$, relative to endpoints, and similarly, let $E$ be a homotopy from $\delta$ to $\delta'$. Then define
\[
E \cdot H : I \times I \to X, \quad E \cdot H(t, s) = \begin{cases} 
H(2t, s) & \text{if } t \leq 1/2 \\
E(2t - 1, s) & \text{if } t \geq 1/2
\end{cases}
\]
This gives the required homotopy. \qed

Composition of paths is not associative—you can check directly that $\gamma \cdot (\delta \cdot \epsilon) \neq (\gamma \cdot \delta) \cdot \epsilon$.

Proposition 24.5. Suppose $\gamma, \delta$ and $\epsilon$ are paths in $X$ such that $\gamma \cdot (\delta \cdot \epsilon)$ is defined. Then $[\gamma][\delta][\epsilon] = [\gamma][\delta][\epsilon]$.

That is, the composition is associative once we pass to homotopy classes.
**Proof.** It is sufficient to write down a homotopy (relative to endpoints) between $\gamma \cdot (\delta \cdot \epsilon)$ and $(\gamma \cdot \delta) \cdot \epsilon$.

$$H(t, s) = \begin{cases} 
\gamma(2t + 2st) & \text{if } t \leq 1/2 - s/4 \\
\delta(4t - 2 + s) & \text{if } 1/2 - s/4 \leq t \leq 3/4 - s/4 \\
\epsilon(4t - 2st + 2s - 3) & \text{if } t \geq 3/4 - s/4 
\end{cases}$$

**Definition 24.6.** If $X$ is a space and $x \in X$, define $e_x$ to be the constant path at $x$, i.e., $e_x(t) = x$ for all $t$.

**Proposition 24.7.** Let $X$ be a space and let $\gamma$ be a path in $X$ starting at $x$ and ending at $y$. Then $[e_x] \cdot [\gamma] = [\gamma]$ and $[\gamma] \cdot [e_y] = [\gamma]$.

**Proof.** We’ll show one of these. The other is similar.

Just write down a homotopy from $e_x \cdot \gamma$ to $\gamma$.

$$H(t, s) = \begin{cases} 
x & \text{if } 2t \leq 1 - s \\
\gamma(2t - st - 1 + s) & \text{if } 2t \geq 1 - s 
\end{cases}$$

**Notation 24.8.** If $\gamma : I \to X$ is a path, write $\gamma^{rev}$ for the reverse of $\gamma$: $\gamma^{rev}(t) = \gamma(1 - t)$. Clearly, $(\gamma^{rev})^{rev} = \gamma$.

**Proposition 24.9.** In the notation above, if $\gamma$ is a path from $x$ to $y$, then $[\gamma] \cdot [\gamma^{rev}] = [e_x]$.

**Proof.** We write down a homotopy:

$$H(t, s) = \begin{cases} 
\gamma(t) & \text{if } 2t \leq 1 - s \\
\gamma(1 - s) & \text{if } 1 - s \leq 2t \leq 1 + s \\
\gamma(2 - 2t) & \text{if } 1 + s \leq 2t 
\end{cases}$$

**Definition 24.10.** A **groupoid** $\mathcal{G}$ is a category having a set of objects and a set of morphisms and such that all morphisms are isomorphisms.

**Example 24.11.** If a groupoid $\mathcal{G}$ has a unique object $*$, then the data of the groupoid is really just $\text{Mor}_\mathcal{G}(*, *)$. This is a set equipped with an associative composition law and an identity element, and where each element has an inverse. That is, it is a group.

**Definition 24.12.** Let $X$ be a topological space and let $A \subseteq X$ be a subset of $X$. Define a **fundamental groupoid** of $X$ with endpoints in $A$, denoted $\Pi(X, A)$, as the groupoid where

$\text{ob} \Pi(X, A) = A$

and for $a_0, a_1 \in A$, the set of morphisms from $a_0$ to $a_1$, is the equivalence classes of paths in $X$ starting at $a_0$ and ending at $a_1$. The previous propositions ensure that the composition law is well defined and associative, that $[e_{a_0}]$ is the identity at $a_0$ and that inverses exist for all morphisms (just reverse the path). So this really is a groupoid.
Remark 24.13. Two special cases of the above are $\Pi(X, X)$, which is called the fundamental groupoid of $X$, and the case where $A = \{x_0\}$ is a distinguished point.

In this case $\Pi(X, \{x_0\})$ consists of homotopy classes (relative to $\{0, 1\}$) of maps $\gamma : I \to X$, starting and ending at $x_0$. This is the same as homotopy classes of maps $\gamma : S^1 \to X$ (since $S^1 \approx I/\{0, 1\}$), relative to the basepoint of $S^1$, which is $[S^1, X], = \pi_1(X, x_0)$.

So $\Pi(X, \{x_0\}) = \pi_1(X, x_0)$. This is called the fundamental group of $X$ with basepoint $x_0$, and it is indeed a group, since it is a groupoid with one object.

Remark 24.14. If $f : X \to Y$ is a continuous map of spaces, and if $A \subseteq B$, then there is an induced map of groupoids $f_* : \Pi(X, A) \to \Pi(Y, f(A))$, given by sending a point $a$ to $f(a)$ and the class of a path $\gamma : I \to X$ to the class of $f \circ \gamma$. We have already established enough about relative homotopy and maps to deduce that $f_*$ is well defined and that $(f \circ g)_* = f_* \circ g_*$ whenever $f, g$ are composable maps, and that $(id_X)_*$ is the identity map on groupoids.

Remark 24.15. The term pair, unless otherwise indicated, means a topological space $X$ and a subspace $A$, denoted $(X, A)$. There is a category of pairs, Pair, where the objects are the pairs and a morphism $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$.

We have constructed a functor

$$\Pi : \text{Pair} \to \text{Groupoid}$$

Proposition 24.16. Suppose $f, g : (X, A) \to (Y, B)$ are maps of pairs and that $f \approx g$ relative to $A$. Then $f_* = g_* : \Pi(X, A) \to \Pi(Y, B)$.

Remark 24.17. In order to write down what happens when the homotopy is not relative to $A$, which is the more interesting case, we have to define the concept of a natural transformation. We may do this later, if there's time.

25 Covering Spaces

Definition 25.1. A map of topological spaces $f : Y \to X$ is a covering space map if, for all $x \in X$, there is some open $U \ni x$ such that the inverse image $f^{-1}(U)$ is homeomorphic to a disjoint union $\bigsqcup_{j \in J} V_j$ such that each induced map $f|_{V_j} : V_j \to U$ is a homeomorphism.

Remark 25.2. Some people might require the map $f$ to be surjective, but we do not.

Example 25.3. The prototypical examples are $f_u : S^1 \to S^1$ given by $z \mapsto z^u$, and $f_\infty : \mathbb{R} \to S^1$ given by $f_\infty(t) = (\cos 2\pi t, \sin 2\pi t)$.

Example 25.4. Other, more trivial, examples, include $X \sqcup X \to X$ or the inclusion of open component into a disconnected but locally connected space. These are sort of silly, so we generally concentrate in examples on the cases where both $X$ and $Y$ are connected.

Proposition 25.5. A covering space map is open.

Definition 25.6. Let $f : Y \to X$ be a map and let $x \in X$. Define the fibre of $f$ at $x$ to be $f^{-1}(x)$, and denote it $F_x$.

Notation 25.7. Let $f : Y \to X$ be a covering space and let $W \subseteq X$ be an open set. We say that $f$ trivializes over $W$ if $f^{-1}(W)$ is a disjoint union of open sets mapping homeomorphically to $W$.

Here comes a technical and very important proposition.
Proposition 25.8. Suppose \( f : Y \to X \) is a covering map and that \( Z \) is a space and \( i_0 : Z \to Z \times I \) is inclusion at 0 and that there are maps as indicated making the diagram (without the dashed arrow) commute:

\[
\begin{array}{ccc}
Z & \xrightarrow{G_0} & Y \\
\downarrow{i_0} & & \downarrow{f} \\
Z \times [0, 1] & \xrightarrow{g} & X
\end{array}
\]

Then there is a unique map \( G \) making both triangles commute.

Proof. Let \( \mathcal{W} \) be an open cover of \( X \) trivializing \( f \). For each \( z \in Z \), we can find a cover of \( z \times I \) by open sets \( g^{-1}(W) \) where \( W \in \mathcal{W} \). We may choose a finite subcover: \( g^{-1}(W_1) \cup g^{-1}(W_2) \cup \cdots \cup g^{-1}(W_N) \), and by the tube lemma, we can find an open \( U \subseteq Z \) such that \( z \in U \) and

\[
U \times I \subseteq \bigcup_{i=1}^{N} g^{-1}(W). 
\]

We may even refine each set \( U \times I \cap g^{-1}(W_i) \) into disjoint sets of the form \( U \times (a, b) \), and sets like this cover \( \{z\} \times I \), so that we may take a finite subcover of sets

\[
U \times [0, a_1), U \times (a_2, a_3), U \times (a_4, a_5), \ldots
\]

From there we can produce finitely many values \( 0 = t_0, \ldots, t_M = 1 \) such that the sets \( \{U \times [t_i, t_{i+1}]\}_{i=0}^{M-1} \) is a (not open) cover of \( \{z\} \times I \) with the property that \( g(U \times [t_i, t_{i+1}]) \subset W \) for some \( W \in \mathcal{W} \).

It is now ‘easy’ to show that there is a unique lift \( G_{ij} \) on \( U \times I \). To do this, we simply note that for diagrammatic reasons, there is a unique lift on \( U \times [t_0, t_1] \), and then proceed by induction.

We have now produced continuous lifts \( G_{ij} \) on open subsets \( U \times I \) of \( Z \times I \). Two lifts \( G_{ij} \) and \( G_{ij'} \) agree on \( U \cap U' \). Therefore there is a uniquely defined set map \( G : Z \times I \to Y \), and it is now an exercise to show that \( G \) is continuous.

This allows us to make the following construction.

Construction 25.9. Let \( X \) be a topological space and \( A \subseteq X \). Let \( f : Y \to X \) be a covering space. For each \( a \in A \), define \( F_a = f^{-1}(a) \), which is a discrete topological space, i.e., a set. For each path \( \gamma \) in \( X \) from \( a \) to \( b \), define \( \tilde{\gamma} : F_a \to F_b \) by constructing the unique lift in

\[
\begin{array}{ccc}
\{0\} & \xrightarrow{a} & Y \\
\downarrow{\Gamma} & & \downarrow{\gamma} \\
I & \xrightarrow{\gamma} & X
\end{array}
\]

and letting \( \tilde{\gamma} = \Gamma(1) \).

Proposition 25.10. The map \( \tilde{\gamma} \) defined above depends only on \( [\gamma] \), the homotopy class of \( \gamma \) (rel. endpoints).

Proof. Suppose we have two paths \( \gamma \sim \delta \) in \( X \) (the homotopy being relative to endpoints). We can write down the homotopy between these by defining a function \( H : I \times I \to X \) that is \( \gamma \) on \( I \times \{0\} \) and is a rescaling of \( \delta \) on the other three edges. Then uniqueness of lifting ensures that the lift \( \Gamma \) is homotopic (relative to endpoints) to \( \Delta \), the lift of \( \delta \). In particular, the endpoints are the same. \( \square \)
Proposition 25.11. For any \( A \subseteq X \), the construction assigning to \( a \in A \) the set \( F_a \) and to any \( [\gamma] \) the map \( \tilde{\gamma} \) as defined above is a functor \( F : \Pi(X, A)^{op} \to \text{Set} \).

Proof. We need only check that the lift of constant paths are identity maps (easy) and that \( \tilde{\gamma \delta} = \delta \tilde{\gamma} \), which follows from uniqueness of lifts.

26 Basepoints and the fundamental group

From here on, to avoid getting bogged down in category theory, we restrict ourselves to the special case \( A = \{ x_0 \} \), so that \( \Pi(X, A) = \pi_1(X, x_0) \).

If we are going to work with this seriously, we have to pay attention to basepoints. The following are useful results.

Proposition 26.1. Let \( X \) be a topological space, let \( x_0 \) and \( x_1 \) be points in \( X \) and let \( \alpha \) be a path from \( x_0 \) to \( x_1 \). There is an isomorphism \( \phi_\alpha : \pi_1(X, x_0) \to \pi_1(X, x_1) \) given by \( \gamma \mapsto \alpha^{-1} \gamma \alpha \).

The proof of this is immediate if we consider \( \pi_1(X, x_0) \) as a subgroupoid of \( \Pi(X) \).

Proposition 26.2. Let \( X \) and \( Y \) be two spaces and let \( f, g : X \to Y \) be two maps and let \( H : X \times I \to Y \) be a homotopy between them (basepoint free). Let \( x_0 \in X \). Let \( \alpha \) be the path \( t \mapsto H(x_0, t) \), from \( f(x_0) \) to \( g(x_0) \), and let \( \phi_\alpha \) be as above. Then the diagram

\[
\begin{array}{ccc}
\pi_1(Y, f(x_0)) & \rightarrow & \pi_1(X, x_0) \\
g_* & \equiv & \phi_\alpha \\
f_* & \rightarrow & \pi_1(Y, g(x_0))
\end{array}
\]

commutes.

Proof. Let \( \gamma \) be a loop in \( X \) based at \( x_0 \). There are two ways of producing a loop in \( Y \), based at \( g(x_0) \). First, one can produce \( g_* (\gamma) \). Second, one can produce \( \alpha^{-1} f_* (\gamma) \alpha \). The proposition asserts that these two loops are homotopic (relative to endpoints). Writing down the homotopy using \( H \) is left as an exercise.

Corollary 26.3. Suppose \( f : X \to Y \) is a homotopy equivalence and \( x_0 \in X \). Then \( f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \) is an isomorphism.

Notation 26.4. A space \( X \) is simply connected if \( \pi_0(X) \) consists of a single path component and \( \pi_1(X, x_0) = \{ e \} \) for one, and therefore for any, basepoint \( x_0 \in X \).
27 The fundamental group and covering spaces

Proposition 27.1. Let \((X, x_0)\) be a pointed space and let \(f : Y \to X\) be a covering space map. Let \(F = f^{-1}(x_0)\). The functor constructed in 25.11 induces a right \(\pi_1(X, x_0)\)-action on \(F\).

Proof. We define a map \(F \times \pi_1(X, x_0) \to F\) as follows: for \(x \in F\) and \([\gamma] \in \pi_1(X, x_0)\), \(x \cdot [\gamma]\) (or \(x \cdot \gamma\)) is the element obtained by lifting \(\gamma\) to \(\Gamma : I \to Y\), with \(\Gamma(0) = x\), then defining \(x \cdot \gamma = \Gamma(1)\).

Proving that this is a right group action is just a matter of shuffling through the algebra. \(\square\)

Notation 27.2. This action of \(\pi_1(X, x_0)\) on the fibre over \(x_0\) is called the monodromy action.

Remark 27.3. There is a category of covering spaces of \(X\) where the objects are maps \(f : Y \to X\) and the maps are maps

\[
\begin{array}{ccc}
Y & \to & Y' \\
\downarrow f & & \downarrow f' \\
X & \to & X
\end{array}
\]

making the diagram commute. Write this as \(\text{Cov}_X\).

There is an obvious category of right \(\pi_1(X, x_0)\)-sets, written \(\text{Set} - \pi_1(X, x_0)\).

Proposition 27.4. Let \(h : Y \to Y'\) be a map of covering spaces of \((X, x_0)\). Let \(F, F'\) be the fibres of \(Y, Y'\) over \(x_0\) respectively. Then the induced map \(h : F \to F'\) is a map of right \(\pi_1(X, x_0)\)-sets.

Proof. This follows from uniqueness of lifts of paths. \(\square\)

Corollary 27.5. Let \((X, x_0)\) be a based topological space. The taking the fibre at \(x_0\) is a functor

\[F_{x_0} : \text{Cov}_X \to \text{Set} - \pi_1(X, x_0)\]

Proof. This requires us to check identities and compositions. The first are immediate, the second follow from uniqueness of lifts. \(\square\)

Proposition 27.6. Suppose \(f : (Y, y_0) \to (X, x_0)\) is a covering space. Then \(f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)\) is injective and the \(\pi_1(X, x_0)\)-orbit of \(y_0 \in F_{x_0}\) is isomorphic to \(\text{im}(f_*) \backslash \pi_1(X, x_0)\) as right \(\pi_1(X, x_0)\)-sets.

Proof. Suppose \(\gamma\) is a loop in \(Y\) based at \(y_0\), and that \(f_* (\gamma) = e\) in \(\pi_1(X, x_0)\). Then we can implement the homotopy showing \(f_* (\gamma)\) is trivial as a map \(H : I \times I \to X\) such that \(H(t, 0) = f(\gamma)\) and \(H(0, s) = H(1, s) = H(t, 1) = e_{x_0}\). There exists a unique lift of \(H\) to \(\tilde{H} : I \times I \to Y\), and by uniqueness, we know that \(\tilde{H}(0, s) = \tilde{H}(1, s) = \tilde{H}(t, 1) = e_{y_0}\). This shows that \(f_*\) is injective.

The second statement is equivalent, by the orbit–stabilizer theorem, to the assertion that \(\text{im}(f_*)\) is exactly the stabilizer of \(y_0\). But \(\text{im}(f_*)\) is precisely the set of loops in \(X\) that lift to loops based at \(y_0\), i.e., the loops that act trivially on \(y_0\). \(\square\)

Corollary 27.7. Let \(f : Y \to X\) be a covering space, let \(y_0 \in Y\) and \(x_0 \in X\) be basepoints and suppose \(X\) is path connected. Then \(Y\) is path connected if and only if the monodromy action on \(F_{x_0}(Y)\) is transitive.
Proof. If \( Y \) is path connected, then for any \( y \in F_{x_0}(Y) \), we can find a path from \( y_0 \) to \( y \). This descends to a loop \( \delta \) in \( X \) based at \( x_0 \), and by uniqueness of lifting, we see that \( y_0 \delta = y \), so the monodromy action is transitive.

Conversely, assume the monodromy action is transitive. Let \( y \) be any point in \( Y \). There exists some path \( \gamma \) from \( f(\gamma) \) to \( x_0 \), and there exists some lift of this path from \( y \) to a point \( z \in F_{x_0} \). Then the transitivity of the monodromy action means we can find a path from \( z \) to \( y_0 \). By composition, there is a path from \( y \) to \( y_0 \).

Now comes a digression into functors.

**Definition 27.8.** Let \( F : C \rightarrow D \) be a functor. The functor \( F \) is said to be **faithful** if the induced maps

\[
\text{Mor}_C(x, y) \rightarrow \text{Mor}_D(F(x), F(y))
\]

are injective for all \( x, y \in C \). The functor \( F \) is **full** if this map is surjective for all \( x, y \in C \). The functor \( F \) is **essentially surjective** if, for all \( d \in D \), there exists some \( c \in C \) such that \( F(c) \cong d \).

A full, faithful and essentially surjective functor \( F : C \rightarrow D \) is an equivalence of categories.

**Definition 27.9.** A space \( X \) is **locally path connected** if each point \( x \in X \) has a local base \( \{U_j\}_{j \in J} \) consisting of path connected spaces. The space \( X \) is **semilocally simply connected** if each point \( x \in X \) has a local base \( \{U_j\}_{j \in J} \) consisting of sets such that every loop in \( U_j \) based at \( x \) can be contracted to the constant loop at \( x \) through a homotopy in \( X \).

**Theorem 27.10.** Let \( X \) be a connected and locally path connected space. Let \( x_0 \in X \) be a point. Then the functor \( F_{x_0} : \text{Cov}_X \to \text{Set} - \pi_1(X, x_0) \) is full and faithful.

If \( X \) is also semilocally simply connected, then \( F_{x_0} \) is an equivalence of categories.

**Proof.** First we establish fidelity. Suppose

\[
\begin{array}{ccc}
Y & \xrightarrow{g_1} & Z \\
\downarrow{f} & & \downarrow{h} \\
X & \xrightarrow{g_2} & Z
\end{array}
\]

are two maps of covering spaces of \( X \) such that \( F_{x_0}(g_1) = F_{x_0}(g_2) \). Let \( y \in Y \) be a point, and let \( \gamma \) be a path in \( X \) from \( f(\gamma) \) to \( x_0 \). We can lift \( \gamma \) uniquely to a path in \( Y \) ending at \( y \), and taking the starting point of this path gives us \( y_0 \in Y \), a basepoint for \( Y \), and a lift \( \tilde{\gamma} \) of \( \gamma \) to a path from \( y_0 \) to \( y \). By hypothesis, \( g_1(y_0) = g_2(y_0) \), and therefore by uniqueness of lifts in \( Z \), we see that \( g_1(\tilde{\gamma}) = g_2(\tilde{\gamma}) \), so that in particular \( g_1(y) = g_2(y) \). (Note that this requires only path connectedness of the base).

Second we establish fullness. Let \( Y, Z \) be two covering spaces, and let \( \phi : F_{x_0}(Y) \rightarrow F_{x_0}(Z) \) be a map of \( \pi_1(X, x_0) \)-sets. We construct a function \( g : Y \rightarrow Z \) such that \( F_{x_0}(g) = \phi \) as follows. For any \( y \in Y \), we can find a path \( \gamma \) from \( y \) to some element \( y_0 \) of \( F_{x_0}(Y) \). The map \( \phi \) gives us an element \( \phi(y_0) \) and we can lift \( f(\gamma) \) to a path \( \tilde{\gamma} \) in \( Z \) starting at \( \phi(y_0) \). Define \( g(y) = \tilde{\gamma}(1) \).

Ostensibly this depends on our choice of \( \gamma \), which also affects the choice of \( y_0 \), but the fact that \( \phi \) is a map of \( \pi_1(X, x_0) \)-sets means that \( g(y) \) is independent of choices. Two paths \( \gamma \) and \( \gamma' \) descend to a loop \( \delta = \gamma' \gamma^{rev} \) in \( X \). The path \( \gamma' \) gives us \( y_0 \in F_{x_0}(Y) \) and the path \( \gamma' \) gives us \( y'_0 \). Say then \( y_0 \delta = y'_0 \), and since \( \phi \) is a map of \( \pi_1(X, x_0) \)-sets, \( \phi(y_0) \delta = \phi(y'_0) \), which implies (by unique lifting) that the lifts of \( f(\gamma') \) and \( f(\gamma^{rev}) \) compose to give a path from \( y'_0 \) to \( y_0 \), which implies that \( f(\gamma') \) and \( f(\gamma) \) have the same endpoint.
We sketch an argument to show that \( \gamma \) is continuous. Consider \( \gamma \in Y \). Choose an open neighbourhood \( W \) of \( f(\gamma) \in X \) that is path connected and such that both \( f \) and \( h \) trivialize over \( W \). There exists some path connected \( U = W \) that is an open neighbourhood of \( \gamma \in Y \), and some \( V = W \) that is an open neighbourhood of \( g(\gamma) \in Z \). The argument showing that \( g(\gamma) \) is independent of the choice of \( \gamma \) then shows that \( g|U : U \to Z \) maps \( U \) homeomorphically to \( V \). (Note that this argument required only path connectedness and local path connectedness).

We establish essential surjectivity. Let \( S \) be a right \( \pi_1(X, x_0) \)-set. It suffices to show that \( S \) is isomorphic to \( F_{s_0}(Y) \) for some connected \( Y \) whenever \( S \) consists of a single orbit, because we can decompose a general \( S \) as a disjoint union of orbits and a general covering space as a disjoint union of connected covering spaces.

So assume \( S \) consists of a single orbit. If \( S \) is empty, there is nothing to do. Let \( s_0 \in S \) be a basepoint, therefore. Consider \( K \), the stabilizer of the \( \pi_1(X, x_0) \) action on \( s_0 \). This is a subgroup of \( \pi_1(X, x_0) \), and \( S \) is isomorphic, as a right \( \pi_1(X, x_0) \)-set, to \( K/\pi_1(X, x_0) \).

Now impose an equivalence relation on paths \( \gamma : I \to X \); we say \( \gamma \sim \gamma' \) if \( \gamma' \circ (\gamma')^{-1} \in K \). Let \( E \) denote the set of equivalence classes of such paths and denote the class of \( \gamma \) by \( [\gamma] \). We construct a topological space \( Y \) as follows. As a set, \( Y = X \times E \), and there is an obvious function \( f : Y \to X \).

We generate a topology on \( Y \) as follows. Choose a point \((x, [\gamma])\). Find \( W \ni x \) that is both path connected and semilocally simply connected—any \( x \) has a local base consisting of such sets. For each \( x' \) in \( W \), there exists a path \( \delta \) from \( x \) to \( x' \). The class of \([\gamma\delta]\) is independent of the choice of \( \delta \) by virtue of semilocal simply-connectedness: any alternative choice of \( \delta' \) would give us \( \delta'\delta^{-1} = e_x \). Define an open set \( W_{y} \subseteq Y \) to consist of

\[
\{(x', [\gamma\delta]) \mid y \in Y, x' \in W, \delta : I \to W\}.
\]

With this topology, \( f : Y \to X \) is continuous, and by restricting to open sets \( W \) that are locally path connected and semilocally simply connected, we see that \( f \) is a covering space map. Finally, \( F_{s_0}(Y) \) is isomorphic to \( K/\pi_1(X, x_0) \) as a right \( \pi_1(X, x_0) \)-set, as required.

**Example 27.11.** Let \((X, x_0)\) be a pointed space meeting all the conditions of the theorem and \( \pi_1(X, x_0) \) its fundamental group. There exists a covering space \( f : \tilde{X} \to X \) corresponding to \( \pi_1(X, x_0) \) viewed as a set over itself with action by right multiplication. Since the action is transitive, the space \( \tilde{X} \) is connected, and since the action is free, for any choice of basepoint \( \tilde{x}_0 \) over \( x_0 \), the fundamental group \( \pi_1(\tilde{X}, \tilde{x}_0) \) is trivial (its injective image corresponds to the stabilizer of a point in \( \pi_1(X, x_0) \)).

This covering space \( f : \tilde{X} \to X \) is distinguished up to isomorphism of covering spaces by being simply connected—just as \( \pi_1(X, x_0) \) is distinguished among \( \pi_1(X, x_0) \)-sets by having a free transitive action. It is called the universal cover of \( X \).

If \( f : \tilde{X} \to X \) is the universal cover of \( X \), then \( F_{s_0} \) is isomorphic to \( \pi_1(X, x_0) \) as a right \( \pi_1(X, x_0) \)-space.

**Example 27.12.** The universal cover of \( S^1 \) is \( f : \mathbb{R} \to S^1 \) given by \( f(t) = (\cos(2\pi t), \sin(2\pi t)) \). The fundamental group \( \pi_1(S^1, s_0) \) is identified with \([0, 1, 2, \ldots] = F \) as a set with a right \( \pi_1(S^1, s_0) \)-action. One can see directly by lifting that the element 1 acts on \( n \in F \) by \( n \mapsto n + 1 \), whereupon we see that \( \pi_1(S^1, s_0) \cong (\mathbb{Z}, +) \).

**Example 27.13.** Similarly, let \( \Gamma_n \) denote an infinite tree in which each vertex has degree \( 2n \). Assuming for the moment the true statement that all trees are contractible, \( \Gamma_n \) is the universal cover of the bouquet of \( n \)-circles \( B = S^1 \vee S^1 \vee \cdots \vee S^1 \). Arguing similarly to the circle case, we see that \( \pi_1(B, b_0) \) is a free group on \( n \) letters.

It is instructive even in the case \( n = 2 \) to consider what \( \pi_1(B, b_0) \)-sets are. They are sets equipped with two bijective functions. One can take any such set and produce a covering space of \( B \), and vice versa.
28 Deck transformations

Definition 28.1. Let \( f : Y \to X \) be a covering space. An automorphism \( h : Y \to Y \) of \( Y \) over \( X \) is called a \emph{deck transformation}. The set of all such transformations forms a group, \( \text{Aut}_X(Y) \).

Definition 28.2. Let \( K \subseteq G \) be a subgroup of a group. Then \( N_G(K) \), the normalizer of \( K \) in \( G \) is the subgroup of elements \( g \in G \) such that \( gKg^{-1} = K \).

Lemma 28.3. Let \( K \subseteq G \) be a subgroup of a group. Then the group of automorphisms of \( K \backslash G \) as a right \( G \)-set is canonically identified with \( N_G(K)/K \).

Proof. The identification is by means of \( Kg \mapsto KnG \) where \( n \in N_G(K) \).

Proposition 28.4. Let \( X \) be connected and locally path connected. Let \( x_0 \in X \) be a basepoint and write \( G = \pi_1(X, x_0) \). Let \( f : Y \to X \) be a covering space such that \( Y \) is connected and has basepoint \( y_0 \) with \( f(y_0) = x_0 \). Write \( K \) for the image of the fundamental group of \( Y \) in \( G \). Then the group of deck transformations of \( Y \) over \( X \) is canonically identified with \( N_G(K)/K \).

Proof. Write \( G = \pi_1(X, x_0) \), and work in the equivalent category of right \( G \)-sets, by means of the functor \( F_{x_0}(Y) \). A deck transformation is then nothing more than an automorphism of \( F_{x_0}(Y) \) as a set with right \( G \)-action.

Remark 28.5. The deck transformations are functions, and therefore compose as functions do, so that \( N_G(K)/K \) acts on \( Y \) on the left.

Remark 28.6. Retain the hypotheses imposed above on \( X \) and \( Y \). Since the deck transformations act on \( Y \) over \( X \), they act in particular on \( F_{x_0}(Y) = f^{-1}(x_0) \). This action is a left action, and is an action by \( N_G(K) \), which is, in general, a different group from \( G = \pi_1(X, x_0) \).

If we allow the covering space \( Y \) to be disconnected, the group of deck transformations can become larger than \( \pi_1(X, x_0) \). For instance, the split covering \( * \coprod * \to * \) has nontrivial deck transformation group.

In general, the left action by deck transformations is different from the right monodromy action by \( \pi_1(X, x_0) \).

Definition 28.7. Let \( f : Y \to X \) be a covering space where \( Y \) is connected and locally path connected. Let \( y_0 \in Y \) be a basepoint mapping to \( x_0 \in X \). We say \( f : Y \to X \) is a \emph{normal} covering space (for the basepoints \( y_0 \) and \( x_0 \) if the group of deck transformations \( \text{Aut}_X(Y) \) acts transitively on \( F_{x_0} \).

Proposition 28.8. Assume \( f : Y \to X \) is a covering space where \( Y \) is connected and locally path connected. The following are equivalent:

1. The covering is normal for one choice of basepoints \( y_0 \in Y \) and \( x_0 \in X \);
2. The subgroup \( f_* (\pi_1(Y, y_0)) \subseteq \pi_1(X, x_0) \) is normal;
3. The covering is normal for all choices of basepoint.

Proof. The action by \( N_G(K)/K \) is transitive on the fibre \( K \backslash G \) if and only if \( N_G(K) = G \), which is to say, if \( K \) is normal in \( G \).

The inclusion \( f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0) \) is isomorphic to the inclusion \( f_* : \pi_1(Y, y'_0) \to \pi_1(X, x'_0) \) given by a different choice of basepoints.

Remark 28.9. Let \( f : \tilde{X} \to X \) be a universal cover of a connected, locally path connected, semilocally simply connected space. Then \( \text{Aut}_X(\tilde{X}) \cong \pi_1(X, x_0) \).