Chapter 6

Connectedness

20  Connectedness

Definition 20.1. We say a topological space $X$ is connected if every function $X \to D$ where $D$ has a discrete topology is constant.

Lemma 20.2. Let $X$ be a topological space. The following are equivalent:

1. $X$ is connected;
2. Every function $X \to \{0,1\}$, where the target is discrete, is constant.
3. If $A \subseteq X$ is open and closed, then $A = \emptyset$ or $A = X$.

Example 20.3. Let $A \subseteq \mathbb{R}$ be a subset of the real line. If $x < y < z$ are three points in $\mathbb{R}$ such that $x, z \in A$ but $y \notin A$, then the function $f : \mathbb{R} \setminus \{y\} \to \{0,1\}$ given by $f(t) = 0$ if $t < y$ and $f(t) = 1$ otherwise is continuous and so restricts to a continuous nonconstant function on $A$. So $A$ is not connected (it is disconnected).

This implies that if $A$ is a connected subset of $\mathbb{R}$, then $A$ is an interval (including the degenerate intervals $\emptyset$ and $\{a\}$). On the other hand, if $A$ is an interval and $f : A \to \mathbb{R}$ has the property that $f(x) = 0$ and $f(z) = 1$ (wlog $x < z$) and that $f$ is continuous, then consider the element $y = \inf(f^{-1}(1) \cap [x,z])$. Since $f^{-1}(1)$ is closed, $f(y) = 1$, but since $f^{-1}(0)$ is closed, $f(y) = 0$, a contradiction.

It follows that the connected subsets of $\mathbb{R}$ are precisely the intervals.

Proposition 20.4. Let $f : X \to Y$ be a continuous surjective function, and suppose $X$ is connected. Then $Y$ is connected.

Proposition 20.5. Let $X$ be a topological space, let $\{A_i\}_{i \in I}$ be a family of connected subspaces of $X$ such that for all $i, j \in I$, the set $A_i \cap A_j$ is nonempty and such that $\bigcup_{i \in I} A_i = X$. Then $X$ is connected.

Corollary 20.6. Let $\{X_i\}_{i \in I}$ be a family of connected spaces. Then $X = \coprod_{i \in I} X_i$ is connected.

Proof. If any of the sets $X_i$ is empty, then the product is empty and there is nothing to do.

Let $f : X \to \{0,1\}$ be a continuous function that is not identically 0. First we observe that if $y$ and $y'$ are two points of $X$ that differ only in one coordinate, the $j$-th coordinate, then $f(y) = f(y')$. This is because there is an inclusion $c_j : X_j \to \prod_{i \in I} X_i$ given by the other coordinates, and the composite $f \circ c_j$ gives a continuous function $X_j \to \{0,1\}$.
By an easy induction, if \( y \) and \( y'' \) differ in finitely many coordinates, then \( f(y) = f(y'') \).

Finally, suppose \( x \in X \) is such that \( f(x) = 1 \). Then \( f^{-1}(1) \) contains a subbasic open set \( U \) around \( x \), so that there are finitely many coordinates \( i_1, \ldots, i_r \) such that if \( x \) and \( y'' \) agree in these coordinates, then \( y'' \in U \). In particular, \( f(y'') = 1 \) as well. But then for arbitrary \( y \in X \), we can change finitely many coordinates to produce \( y'' \in U \), so \( f(y) = 1 \). Therefore \( f \) is identically 1.

**Definition 20.7.** Let \( X \) be a topological space and let \( x \in X \) be point. The connected component \( C_x \) of \( x \) is the union of all connected subsets of \( X \) containing \( x \).

**Remark 20.8.** The connected components are connected, and if \( y \in C_x \), then \( C_y = C_x \).

**Example 20.9.** The space \( Q \) shows that the connected components are not necessarily open and closed. A space in which connected components are singletons is said to be **totally disconnected**.

The space \( Q \) seems badly behaved from a certain point of view: one might like the connected components themselves to be both closed and open subsets of the space, but this is not the case.

**Remark 20.10.** Every topological space is a union of connected components, and these components are pairwise disjoint.

**Definition 20.11.** A space \( X \) is **locally connected** if every point \( x \in X \) has a local base \( \{U_i\}_{i \in I} \) such that \( U_i \) is connected.

**Proposition 20.12.** Let \( X \) be a locally connected space, and let \( X_\alpha \) be the connected component of \( x \in X \). Then \( X_\alpha \) is open and closed.

**Proof.** It suffices to prove \( X_\alpha \) is open, since then the complement is the union of the other components of \( X \), which is also open.

The component \( X_\alpha \) contains open neighbourhoods around each point, and is therefore open.

## 21 Path-connectedness

Let \( I \) denote \([0,1]\) with the usual topology throughout.

**Definition 21.1.** A topological space \( X \) is **path-connected** if for every two points \( x, y \in X \), there is a continuous function \( \gamma: I \to X \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \).

**Proposition 21.2.** If \( X \) is a topological space and \( \{A_i\} \) is a set of path connected subsets that cover \( X \) and such that \( \bigcap A_i \neq \emptyset \), then \( X \) is path connected.

**Corollary 21.3.** If \( X \) is path connected, then it is connected.

**Proposition 21.4.** If \( \{X_i\}_{i \in I} \) is a family of path-connected topological spaces, then \( X = \prod_{i \in I} X_i \) is path-connected.

**Definition 21.5.** A **path component** of \( X \) is a maximal path-connected subspace.

**Remark 21.6.** As in the case of connected components, the relation "\( x \sim y \) if \( x \) and \( y \) are in the same path component" is an equivalence relation on \( X \).

**Definition 21.7.** A space \( X \) is **locally path connected** if every point has a local base consisting of path connected neighbourhoods.
Proposition 21.8. Suppose $X$ is a connected locally path-connected space. Then $X$ is path-connected.

Proof. As in the case of connectedness, the locally path-connected hypothesis implies that path-components are open and closed. $X$ can have no nontrivial open-and-closed sets, so $X$ has a single path-component.

Example 21.9. The topologist’s sine curve $S$ is a well-known metric space that is connected but not path connected. This set is defined as

$$S = \{(0, y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin \frac{1}{x}) \mid x > 0\}.$$ 

First we show that this is connected. The subset $C = \{(x, \sin \frac{1}{x}) \mid x > 0\}$ is homeomorphic to $(0, \infty)$ and is therefore connected. Let $(0, t)$ be a point in $L = \{(0, y) \mid -1 \leq y \leq 1\}$, and let $f : S \to \{0,1\}$ be a continuous function taking (w.l.o.g.) the value 0 on $C$. Then there is a sequence in $C$ converging to $(0, t)$, so that $f((0, t)) = 0$ by continuity. So the function $f$ is constant. This shows $S$ is connected.

Now we show that $S$ is not path connected. Suppose $f : I \to S$ is a path from a point in $L$ to $p = (1/\pi, 0)$. Consider the set $f^{-1}(L)$, which is closed in $L$, and therefore has a maximal element, $t_0 < 1$. Restricting, we have found a continuous function $f : [t_0, 1] \to S$ with the property that $f(t_0) = (0, y_{t_0})$ and $(x_t, y_t) := f(t) \in C$ for $t > t_0$. Since $f$ is continuous, $\lim_{t \to t_0^+} x_t = 0$, and by the intermediate value theorem we can find sequences of values of $t$

$$(t_n^+) \text{ such that } x_{t_n^+} = \frac{1}{2n\pi}$$

and

$$(t_n^-) \text{ such that } x_{t_n^-} = \frac{3}{2n\pi}$$

each converging to $t_0$. But then $f(t_n^+) = (t_n^+, 1)$ while $f(t_n^-) = (t_n^-, -1)$, so that the $y$ coordinate of $f(t_0)$ must be both $\lim_{n \to \infty} 1$ and $\lim_{n \to \infty} -1$. 