Chapter 5

The Compact–Open Topology

18 Definition

The fundamental problem in this section is how to put a topology on the set of all continuous functions \( f : X \to Y \). In good circumstances, which is to say when \( X \) is locally compact and Hausdorff, there is a standard default choice, which is the subject of this chapter.

The subject of functional analysis is the study of topologies and metrics on spaces of functions, and we know that functional analysis is not a small topic, so this chapter is not the last word on spaces of functions, by any means.

Notation 18.1. Let \( X \) and \( Y \) be topological spaces. Let \( K \subseteq X \) be compact and \( U \subseteq Y \) be open. Write \( \mathcal{o}(K, U) \) for the set of continuous functions \( f : X \to Y \) such that \( f(K) \subseteq U \).

Definition 18.2. Let \( X \) and \( Y \) be topological spaces. Let \( \mathcal{C}(X, Y) \) denote the set of continuous functions from \( X \) to \( Y \), endowed with the topology generated by open sets of the form \( \mathcal{o}(K, U) \) as above. This topology is called the compact–open topology.

Example 18.3. Suppose \( K \subseteq X \) is a singleton \( \{x\} \). This set is guaranteed to be compact. Let \( U \) be any open set in \( Y \). Then the set \( \mathcal{o}(\{x\}, U) \) is the set of continuous \( f \) such that \( f(x) \in U \).

This has the following consequence for the compact–open topology: if \( (f_n) \to f \) is a convergent sequence in \( \mathcal{C}(X, Y) \), and if \( f(x) \in U \), then each open set \( \mathcal{o}(\{x\}, U) \) has to contain a tail of \( (f_n) \). This is equivalent to saying that some tail \( f_n(x), f_{n+1}(x), \ldots \) is contained in \( U \), or in other other words, that \( (f_n(x)) \to f(x) \).

Example 18.4. Suppose \( Y \) is a metric space, and \( X \) is Hausdorff. In this case the compact-open topology is also called the topology of uniform convergence on compact subsets for the following reason.

A sequence of function \( (f_n) \) in \( \mathcal{C}(X, Y) \) converges to \( f \) in the compact-open topology if and only if, for all compact \( K \subseteq X \) and all \( \epsilon > 0 \), there exists some \( N \in \mathbb{N} \) such that \( d(f_n(k), f(k)) < \epsilon \) for all \( k \in K \).

The “only if” direction is largely an exercise on the homework—the homework asks you to prove that when \( X = K \) is compact, the topology on \( \mathcal{C}(X, Y) \) is metric with the so-called uniform metric. Below we will also show that the map determined by restriction of functions \( \mathcal{C}(X, Y) \to \mathcal{C}(K, Y) \) is continuous.

Let’s do the “if” direction. Suppose \( (f_n) \to f \) uniformly on all compact sets. Consider a general open neighbourhood of \( f \). This contains an open neighbourhood of the form

\[
W = \bigcap_{i=1}^{n} \mathcal{o}(K_i, U_i) \ni f
\]
because opens like this form a basis.

**Proposition 18.5.** The construction of \( \mathcal{C}(X, Y) \) is contravariantly functorial in \( X \) and covariantly functorial in \( Y \). In less technical language, if \( g : X' \to X \) and \( h : Y \to Y' \) are continuous functions, then precomposition with \( g \) and postcomposition with \( h \) yields a function

\[
\Phi_{g,h} : \mathcal{C}(X, Y) \to \mathcal{C}(X', Y') \quad f \mapsto h \circ f \circ g
\]

and this function is continuous.

**Proof.** It is sufficient to show that \( \Phi_{g,h}^{-1}(o(K, U)) \) is open when \( K \) is compact in \( X' \) and \( U \) is open in \( Y' \). This is \( o(g(K'), h^{-1}(U)) \), which is open. \( \square \)

**Proposition 18.6.** Let \( Y \) be a topological space, then the map \( Y \to \mathcal{C}(\ast, Y) \) sending \( y \) to the constant function with value \( y \) is a homeomorphism.

## 19 Currying and Uncurrying

Suppose we have a continuous function \( f : X \times Y \to Z \). We can **curry** this function to produce a function \( \alpha_f : X \to \mathcal{C}(Y, Z) \) defined by

\[
\alpha_f(x)(y) = f(x, y).
\]

The function \( \alpha_f(x) : Y \to Z \) is the composite

\[
Y \xrightarrow{i_x} X \times Y \xrightarrow{f} Z
\]

and since both functions here are continuous, so is \( \alpha_f(x) \).

**Proposition 19.1.** The function \( \alpha_f : X \to \mathcal{C}(Y, Z) \) is continuous.

**Proof.** It suffices to prove that \( \alpha_f^{-1}(o(K, U)) \subseteq X \) is open, where \( K \subseteq Y \) is compact and \( U \subseteq Z \) is open.

Explicitly: \( \alpha_f^{-1}(o(K, U)) \) is the set of all \( x \in X \) such that for all \( y \in K \), the value \( f(x, y) \in U \).

Fix \( K, U \) and suppose \( x \in \alpha_f^{-1}(o(K, U)) \). This implies that \( f(|x| \times K) \subseteq U \) in \( Z \), or equivalently \( |x| \times K \subseteq f^{-1}(U) \). Then by the generalized tube lemma, there are some open \( V \ni x \) and \( W \ni K \) such that \( f(V \times W) \subseteq U \). In particular, \( f(V \times K) \subseteq U \), which implies that \( x \in V \subseteq \alpha_f^{-1}(o(K, U)) \). Since \( x \) was arbitrary, \( \alpha_f^{-1}(o(K, U)) \) is open. \( \square \)

We can also **uncurry** functions. Suppose \( f : X \to \mathcal{C}(Y, Z) \) is a continuous function, then we can define \( \beta_f : X \times Y \to Z \) by the formula \( \beta_f(x)(y) = f(x)(y) \).

**Proposition 19.2.** With the definition as above, if \( Y \) is locally compact and Hausdorff, then \( \beta_f \) is continuous.

**Proof.** Let \( U \) be an open set in \( Z \). We want to show that \( \beta_f^{-1}(U) \) is open. To do this, we take \( (x, y) \in \beta_f^{-1}(U) \) and show that it has some neighbourhood \( W \times V \) contained in \( \beta_f^{-1}(U) \).

The element \( f(x) \in \mathcal{C}(Y, Z) \) is a continuous function, so that \( f(x)^{-1}(U) \) is an open set of \( Y \) containing \( y \). Since \( Y \) is locally compact and Hausdorff, there is some open \( V \) satisfying \( y \in V \subseteq \hat{V} \subseteq f(x)^{-1}(U) \) such that \( \hat{V} \) is compact. Now consider the open set \( o(\hat{V}, U) \) in \( \mathcal{C}(Y, Z) \). It contains \( f \). Furthermore, the set \( W = \)
$f^{-1}(\partial(V, U))$ is open, because $f$ is continuous, and it contains $x$ because $f(x)(\overline{V}) \subseteq f(x)(f^{-1}(U)) = U$. Now if we apply $\beta_f(W \times V)$, we get $f(x')(y')$ where $x' \in W$ and $y' \in V$. Note that $f(x') \in \partial(V, U)$, so that $f(x')(y') \in U$, as required.

**Corollary 19.3.** Let $X, Z$ be topological spaces and $Y$ a locally compact Hausdorff space. The two constructions of currying and uncurrying yield a natural bijective correspondence

\[ \mathcal{C}(X \times Y, Z) \leftrightarrow \mathcal{C}(X, \mathcal{C}(Y, Z)) \]

**Remark 19.4.** If $X$ and $Y$ are both locally compact Hausdorff, then this is actually a homeomorphism.

**Corollary 19.5.** Let $X$ be a locally compact Hausdorff space. Then the evaluation function

\[ ev : \mathcal{C}(X, Y) \times X \to Y \]

given by $ev(f, x) = f(x)$ is continuous.

**Proof.** Apply the previous corollary to the bijection

\[ \mathcal{C}(\mathcal{C}(X, Y) \times X, Y) \leftrightarrow \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Y)). \]

Take the identity function on the right. This corresponds to the evaluation function on the left.