Chapter 4

Urysohn’s Lemma

16 Urysohn’s Lemma

Suppose $X$ is a topological space and suppose that for any two disjoint closed sets $C_0, C_1 \subseteq X$ we can find a continuous $f : X \to \mathbb{R}$ such that $f|_{C_0} \equiv 0$ and $f|_{C_1} \equiv 1$. Then $f^{-1}(-\infty, 1/2)$ and $f^{-1}(1/2, \infty)$ give us two open sets that separate $C_0$ and $C_1$. This implies $X$ is normal.

Theorem 16.1 (Urysohn’s Lemma). Suppose $X$ is a normal topological space and that $C_0$ and $C_1$ are disjoint closed sets in $X$. Then there exists a continuous function $f : X \to [0,1]$ such that $f(C_0) \subseteq [0,0)$ and $f(C_1) \subseteq [1,1]$.

Proof. Use normality to produce a nested sequence of open sets $U_d$, one for each dyadic rational $d = a/2^n$ in $[0,1] \cap \mathbb{Q}$, such that $C_0 \subseteq U_0$ and $C_1 \subseteq X \setminus \overline{U_1}$, and such that $d < d'$ implies $\overline{U_d} \subseteq U_{d'}$. Then define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \notin U_1 \\ \inf_{x \in U_d} d & \text{otherwise} \end{cases}$$

To prove the function $f$ is continuous, let $x \in X$, and fix $\epsilon > 0$. Write $t = f(x)$. We want to find an open neighbourhood $V$ of $x$ such that $f(V) \subseteq (t - \epsilon, t + \epsilon)$. The cases of $t = 0$ and $t = 1$ are exceptional. If $t = 0$, then we can find some $d < \epsilon$ such that $x \in U_d$. Then $U_d = V$ works. If $t = 1$, then we can find $d > 1 - \epsilon$ such that $x \notin \overline{U_d}$. Then $X \setminus \overline{U_d}$ works. Therefore assume $t \in (0,1)$.

We can find some diadic numbers $d_1 > d_2 > t - \epsilon$ such that $x \notin U_{d_1} \supseteq U_{d_2}$. We can also find $d_3 < t + \epsilon$ such that $x \in U_{d_3}$. Then consider $V = U_{d_1} \setminus \overline{U_{d_3}}$. This is an open set containing $x$, and $f(V) \subseteq [d_2, d_3] \subseteq (t - \epsilon, t + \epsilon)$.

So $f$ is continuous.

Example 16.2. Let $X$ be an uncountable space and let $p \in X$ be a point. Define the fortissimo topology on $X$ as follows: a subset $C \subseteq X$ is closed if $p \in C$ or if $C$ is countable.

We claim this space is normal. Two disjoint closed sets consist of two disjoint countable subsets neither containing $p$, or one countable subset and one subset containing $p$. In the first case, the sets are also open since their complements contain $p$. In the second, the countable set not containing $p$ is open, and its complement is also open.

Therefore Urysohn’s lemma applies to $X$. On the other hand, consider a continuous function $f : X \to \mathbb{R}$ such that $f(p) = 0$. Note that $\{p\}$ itself is a closed point. The sets $U_n = f^{-1}((-1/n, 1/n))$ form a countable
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nested family of open sets in $X$, each containing $p$. Therefore each $U_n$ is cocountable. It follows that $f^{-1}(0) = \cap_{n=1}^{\infty} U_n$ is also cocountable, so that in particular, the closed set $\{p\}$ cannot be expressed $f^{-1}(0)$ for any continuous $f : X \to \mathbb{R}$.

**Definition 16.3.** If $X$ is a topological space, a $G_\delta$-set is any subset of $X$ that can be written as an intersection of countably many open subsets.

**Definition 16.4.** A $G_\delta$-space is a topological space $X$ in which every closed subset is a $G_\delta$-set. A space $X$ is perfectly normal if it is normal and a $G_\delta$-space.

**Exercise 16.5.** A space $X$ is perfectly normal if and only if every closed set $C$ is the zero set of a continuous function $f : X \to \mathbb{R}$.

17 Tietze Extension

**Theorem 17.1** (Tietze Extension). Let $X$ be a normal topological space, let $C \subseteq X$ be a closed subspace and let $f : C \to [-1,1]$ be a continuous function. Then there exists a continuous function $F : X \to [-1,1]$ such that $F(c) = f(c)$ for all $c \in C$.

**Remark 17.2.** Of course, the role of $[-1,1]$ can be played by any nontrivial closed interval.

**Proof.** The idea is to produce a sequence of continuous functions $f_n : X \to [-1,1]$ approximating $f$ on $C$.

**Lemma 17.3.** For any $r > 0$, and any continuous function $h : C \to [-r,r]$, there exists a continuous extension $g : X \to [-r/3,r/3]$ such that $|g(a) - h(a)| \leq 2r/3$.

**Proof.** Divide the interval $[-r,r]$ into three equal thirds by closed subintervals: $I_1 \cup I_2 \cup I_3$. Let $C_1 = h^{-1}(I_1)$ and $C_2 = h^{-1}(I_3)$. By Urysohn’s lemma, we can make a continuous function $g : X \to [-r/3,r/3]$ that takes the value $-r/3$ on $C_1$ and $r/3$ on $C_2$. This $g$ has the required property.

Now, the original $f$ satisfies the hypotheses of the lemma with $r = 1$. Define $g_1 : X \to [-1/3,1/3]$ from the lemma. Set $s_1 = g_1$. Now apply the lemma to $f - s_1$, now with $r = 2/3$, to get $g_2$. Apply the lemma to $f - s_2$ with $r = 4/9$ to get $g_3$ and thus $s_3$ and so on.

Now we have produced an infinite sequence of functions $g_n : X \to [-2^{n-1}/3^n,2^{n-1}/3^n]$. By an easy calculus argument (comparison with a geometric series), the sequence

$$F(x) = \sum_{i=1}^{\infty} g_i(x) = \lim_{n \to \infty} s_n(x)$$

converges for all $x \in X$.

A little bit of estimation shows that if $n > m$, then $|s_n(x) - s_m(x)| < (2/3)^m$. In particular, $(s_n(x)) \to F(x)$ uniformly, which is sufficient to prove that $F(x)$ is continuous.

For any $c \in C$, we know that

$$|f(c) - s_n(c)| \leq (2/3)^n$$

so that $f(c) = \lim_{n \to \infty} s_n(c) = F(c)$.

The same estimation as before shows that $F$ takes values in $[-1,1]$. 

\[\square\]
Lemma 17.4. Let $X$ be a normal topological space and let $C$ be a closed subspace. Suppose $f: C \to \mathbb{R}$ is a continuous function, then there exists a continuous function $F: X \to \mathbb{R}$ such that $F(c) = f(c)$ for all $c \in C$.

Proof. Take a homeomorphism $h: \mathbb{R} \to (-1, 1)$. By composing with $h$, we can find $F': X \to [-1, 1]$ such that $F'|_C = h \circ f$. Let $J$ denote the set of points $x \in X$ such that $|F'(x)| = 1$—these points prevent us from applying $h^{-1}$, so we’ll get rid of them. The set $J$ is closed and disjoint from $C$. By Urysohn’s lemma, we can find a continuous $\chi: X \to [0, 1]$ that takes the value 0 on $J$ and 1 on $C$. The function $x \mapsto \chi(x)F'(x)$ is continuous, and takes values in $(-1, 1)$. Consider $F = h^{-1} \circ (\chi F')$. This has all the properties we asked for. □