Chapter 3

Compactness

10 Elementary Theory

Definition 10.1. An open cover \( \mathcal{U} \) of a topological space \( X \) is a collection \( \{ U_i \}_{i \in I} \) of open sets such that \( \bigcup_{i \in I} U_i = X \).

Definition 10.2. A topological space \( X \) is compact if every open cover \( \{ U_i \}_{i \in I} \) contains a finite subcover \( \{ U_1, \ldots, U_n \} \) such that \( X \subseteq \bigcup_{i=1}^n U_i \).

Remark 10.3. Sometimes the term “quasicompact” is used if \( X \) is not Hausdorff. This is the normal usage in algebraic geometry, which is unfortunate since this is also where one sees most non-Hausdorff compact spaces. In French “compact” means what “compact Hausdorff” means in English.

Proposition 10.4. Let \( f : X \to Y \) be a continuous function and suppose \( X \) is compact. Then \( f(X) \) is a compact subspace of \( Y \).

Proof. Let \( \{ U_i \}_{i \in I} \) be an open cover of \( f(X) \). Consider \( \{ f^{-1}(U_i) \}_{i \in I} \), which is an open cover of \( X \), and therefore has a finite subcover \( \{ f^{-1}(U_1), \ldots, f^{-1}(U_n) \} \). The set \( \{ U_1, \ldots, U_n \} \) is the required finite subcover of \( f(X) \). \qed

Proposition 10.5. Suppose \( C \) is a closed subspace of a compact space \( X \). Then \( C \) is compact.

Proof. Consider an open cover \( \{ U_i \}_{i \in I} \) of \( C \). Let \( V_i \) be open in \( X \) and satisfy \( U_i = C \cap V_i \). Consider \( \{ V_i \}_{i \in I} \cup \{ X \setminus C \} \). This is an open cover of \( X \). Any finite subcover induces a finite subcover of \( \{ U_i \} \). \qed

Proposition 10.6. Suppose \( C \) is a compact subspace of a Hausdorff space \( X \). Then \( C \) is closed in \( X \).

Proof. Let \( x \in X \setminus C \). For each \( y \in C \) we can find disjoint open neighbourhoods \( U_y \ni y \) and \( V_y \ni x \). Finitely many of the \( U_y \) suffice to cover \( C \), since it is compact, and therefore there exists an intersection of finitely many \( V_y \) that is disjoint from \( C \). But a finite intersection of open sets is open. This proves that \( x \) has an open neighbourhood disjoint from \( C \). \qed

The following corollary is extremely useful.

Corollary 10.7. Let \( f : X \to Y \) be a continuous bijection between topological spaces where \( X \) is compact and \( Y \) is Hausdorff. Then \( f \) is a homeomorphism.
Proof. We prove that \( f \) is a closed map. This implies that the closed subsets of \( X \) are in bijective correspondence with the closed subsets of \( Y \). Let \( C \subseteq X \) be closed, then \( C \) is compact, so \( f(C) \) is compact, so \( f(C) \) is closed. \( \square \)

**Definition 10.8.** A topological space \( X \) is said to be **sequentially compact** if every sequence has a convergent subsequence.

We will see later that sequential compactness is neither stronger than nor weaker than compactness. For metric spaces, however, we will prove that the two concepts coincide.

### 11 Compactness in Metric Spaces

**Definition 11.1.** A metric space \( X \) is said to be **totally bounded** if, for all \( \varepsilon > 0 \), one can cover \( X \) by finitely many balls \( B(x_i, \varepsilon) \).

**Remark 11.2.** A totally bounded subspace of a metric space is necessarily bounded. Conversely, a bounded subspace of \( \mathbb{R}^n \) is totally bounded. In infinite-dimensional spaces, however, bounded may not imply totally bounded. For instance, the unit ball \( B(0, 1) \) of \( \ell_2 \) is not totally bounded.

**Proposition 11.3.** A compact metric space is totally bounded.

**Proof.** For any \( \varepsilon \), the balls \( B(x, \varepsilon) \) form an open cover. The finite-subcover property implies total boundedness. \( \square \)

**Proposition 11.4.** A compact metric space is complete.

**Proof.** Suppose \( X \) is a compact metric space. There is an isometric embedding \( i : X \to QX \) where \( QX \) is complete and \( i(X) \) is dense. Since \( i(X) \) is compact, it is also closed. Therefore \( i \) is a bijective isometry (a metric equivalence) and \( X \) is complete. \( \square \)

**Proposition 11.5.** A complete, totally bounded metric space \( X \) is sequentially compact. In particular, a compact metric space is sequentially compact.

**Proof.** Let \( X \) be a totally bounded metric space. Let \( (x_n^0) \) be a sequence. We produce a Cauchy sequence recursively. Cover \( X \) by finitely many balls of radius 1. One of these, \( B_1 \), contains a tail of the sequence. Let \( x^1 \) be the sequence that starts with \( x_1^0 \) and thereafter consists of only those terms in \( B_1 \).

Cover \( B_1 \) by finitely many balls of radius 1/2. One of these, \( B_2 \), contains a tail of the sequence \( x^1 \). Form the sequence

\[
x^2 = (x_1^1, x_2^1, \text{subsequent terms in } B_2)
\]

Cover \( B_2 \) by finitely many balls of radius 1/3. One of these, \( B_3 \), contains a tail of the sequence \( x^2 \). Form the sequence

\[
x^3 = (x_1^2, x_2^2, x_3^2, \text{subsequent terms in } B_3)
\]

Note that the sequences \( x^i \) stabilize: \( x_n^i = x_n^{i+1} \) if \( n \leq i \). Form \( x^\infty \) as \( (x_n^0) \). Observe that by virtue of how we constructed this, if \( n < m \), then \( x_n^\infty \) and \( x_m^\infty \) lie in the same \( 1/n \)-ball, so that \( x^\infty \) is a Cauchy subsequence of \( (x_n^0) \).

Since \( X \) is complete, this Cauchy subsequence converges and so \( x^0 \) has a convergent subsequence. \( \square \)
Proposition 11.6. A sequentially compact metric space is complete and totally bounded.

Proof. Suppose $X$ is sequentially compact. If $(x_n)$ is a Cauchy sequence, then $x_n$ converges to the limit of any convergent subsequence. Therefore $X$ is complete.

If $X$ is not totally bounded, we can find some $\varepsilon > 0$ such that $X$ cannot be covered by $\varepsilon$-balls. Therefore there is an infinite sequence $x_n$ of points that are pairwise at distance at least $\varepsilon$ from each other. This sequence can have no Cauchy subsequence and therefore no convergent subsequence, a contradiction.

Theorem 11.7. Let $(X, d)$ be a metric space. The following are equivalent:

1. $X$ is compact,
2. $X$ is complete and totally bounded,
3. $X$ is sequentially compact.

Proof. We have already proved that 1 implies 2 and that 2 is equivalent to 3. Let us now assume that $X$ is sequentially compact (and therefore totally bounded). Let $\{U_i\}_{i \in I}$ be a cover. We claim that for some $n \in \mathbb{N}$, each ball $B(x, 1/n)$ is contained in some $U_i$.

Suppose for the sake of contradiction that this is not the case. Then let $x_n \in X$ be a sequence such that $B(x_n, 1/n)$ is not contained in any $U_i$. The sequence $(x_n)_n$ contains a subsequence converging to some limit $x$. This $x$ is in some $U_i$, and furthermore, there is some $r > 0$ such that $B(x, 2r) \subset U_i$ and such that $B(x, r)$ contains infinitely many terms of $(x_n)$. But then for any $n > 1/r$, we have $B(x_n, 1/n) \subset U_i$, a contradiction.

Therefore the claim holds. Now, fix a radius $1/n$ such that each of the balls $B(x, 1/n)$ is contained in some $U_i$, depending on $x$. Since $X$ is totally bounded, finitely many such balls suffice to cover $X$, and therefore finitely many of the $U_i$s suffice to cover $X$. So $X$ is compact.

Corollary 11.8. A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Corollary 11.9. Let $X$ be a compact topological space and $f : X \to \mathbb{R}$ be a continuous function. Then $f$ attains a maximum value on $X$.

Proof. The image, $f(X)$, is a compact subset of $\mathbb{R}$ and is therefore closed and bounded.

Example 11.10. The quotient $[0,1]/\{0,1\}$ is homeomorphic to $S^1$. We can now prove this quickly. The interval $[0,1]$ is compact, so its surjective image $[0,1]/\{0,1\}$ is also compact. Therefore the map $f : [0,1]/\{0,1\} \to S^1$ given by $(\cos 2\pi \theta, \sin 2\pi \theta)$ is a continuous bijection with compact source and Hausdorff target.

12 Tychanoff’s Theorem

Theorem 12.1 (Tychanoff’s Theorem). Suppose $\{X_i\}_{i \in I}$ is a family of compact topological spaces. Then the product space $\prod_{i \in I} X_i$ is compact.

Remark 12.2. This theorem is equivalent to the axiom of choice (in the generality in which it has been stated). In the case of a product of two compact spaces, $X \times Y$, the theorem is much simpler to prove. This is the most useful form of the theorem, too. But we’ll just do the general version.
Lemma 12.3. Let $X$ be a topological space and suppose that $X$ is not compact. Then there exists a cover $\{U_i\}_{i \in I}$ of $X$ that does not have a finite subcover and that is maximal with this property. I.e., for any open set $V \not\in \{U_i\}_{i \in I}$, the open cover $\{U_i\}_{i \in I} \cup \{V\}$ has a finite subcover.

Proof. We apply Zorn’s lemma. Suppose $\{\mathcal{U}_j\}_{j \in J}$ is a chain of open covers, each without a finite subcover. Then $\mathcal{Y} = \bigcup_{j \in J} \mathcal{U}_j$ is an open cover of $X$. Suppose $\mathcal{Y}$ has a finite subcover $\{U_1, U_2, \ldots, U_n\}$. Then there is some $\mathcal{U}_j$ containing all these sets, a contradiction.

Since any chain of covers-without-finite-subcovers has an upper bound, there must be a maximal such chain by Zorn’s lemma.

Theorem 12.4 (Alexander’s subbase theorem). Let $X$ be a topological space and let $\mathcal{S}$ be a subbase for the topology on $X$, such that $\bigcap \mathcal{S} = X$. The space $X$ is compact if and only if every cover $\{S_i\}_{i \in I} \subseteq \mathcal{S}$ has a finite subcover.

Proof. One direction is trivial: $X$ is compact then a fortiori every subbasic cover has a finite subcover.

Suppose therefore for the sake of contradiction that every cover drawn from $\mathcal{S}$ has a finite subcover, but that $X$ is nonetheless not compact. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a maximal open cover without a finite subcover. Consider $\mathcal{U} \cap \mathcal{S}$. This cannot form a cover of $X$, so there is some $x \in X$ such that $x$ is not contained in any of the sets of $\mathcal{U} \cap \mathcal{S}$. Nonetheless, there exists some $V \in \mathcal{U}$ containing $x$. We can write $x \in S_1 \cap S_2 \cap \cdots \cap S_n \subseteq V$ where $S_i \in \mathcal{S}$ for all $S$. Because of the way we chose $x$, none of the $S_i$ can appear in $\mathcal{U}$, so that each of the covers $\mathcal{U} \cup \{S_i\}$ strictly contains $\mathcal{U}$, and therefore each contains a finite subcover of $X$. These subcovers must involve $S_i$. There are therefore finite subcovers

$$
\{U_{1,1}, U_{1,2}, \ldots, U_{1,m_1}, S_1\}
\{U_{2,1}, U_{2,2}, \ldots, U_{2,m_2}, S_2\}
\vdots
\{U_{n,1}, U_{n,2}, \ldots, U_{n,m_n}, S_n\}
$$

where the open sets $U_{i,j} \in \mathcal{U}$. But then $\bigcup_{i,j} U_{i,j} \cup V$ is a union of open sets in $U$ and it contains every point in $X$, a contradiction.

Proof of Tychonoff’s theorem. The product topology $X = \prod_{i \in I} X_i$ has a subbase given by all sets of the form $\pi_i^{-1}(U)$ where $U$ is open in $X_i$. By virtue of Alexander’s subbase theorem, it is sufficient to prove that any cover of $X$ by sets from this subbase has a finite subcover.

Suppose $\mathcal{S}$ is an open cover of $X$ where the open subsets are taken from the subbase above. For any coordinate $i$, consider the set $\mathcal{S} \cap \{\pi_i^{-1}(U) \mid U \subseteq X_i\}$. We claim that for at least one $i$, the sets $U$ appearing here must form an open cover of $X_i$. Suppose not, then in each $X_i$, there exists some $x_i$ which is not in any of the appropriate open $U \subseteq X_i$. But now consider $x \in X$ such that $\pi_i(x) = x_i$ for all $i$. This does not lie in any set in $\mathcal{S}$, a contradiction. Hence the claim is proved.

We may assume that there is some $X_i$ such that the open sets $\pi_i^{-1}(U_j)$ appearing in $\mathcal{S}$ are such that the $U_j$ cover $X_i$. Since $X_i$ is compact, we can take $U_1, \ldots, U_n$ that cover $X_i$. Then $\pi_1^{-1}(U_1), \ldots, \pi_n^{-1}(U_n)$ cover $X$.

Lemma 12.5 (Generalized tube lemma). Let $X$ and $Y$ be topological spaces and $A \subseteq X$ and $B \subseteq Y$ be compact subsets. If $N$ is an open subset of $X \times Y$ containing $A \times B$, then there exists open subsets $U \subseteq X$ and $V \subseteq Y$ such that $A \times B \subseteq U \times V \subseteq N$. □
Proof. If $A$ is empty, there is nothing to show.

Let $a \in A$. We produce a cover of $a \times B$ as follows. For each $b \in B$, we can find an open set of the form $U_b \times V_b$ such that $(a, b) \in U_b \times V_b \subseteq N$. By compactness, there is a finite set $\{(a, b_1), (a, b_2), \ldots , (a, b_n)\}$ of such points so that the associated $U_{b_1} \times V_{b_1}$ form a cover $A \times B$. If we take $U(a) = \bigcap_{i=1}^{n} U_{b_i}$ and $V(a) = \bigcup_{i=1}^{n} V_{b_i}$, then the open set $U(a) \times V(a)$ has the following properties:

1. It is contained in $N$.
2. It contains $\{a\} \times B$.

We now repeat this procedure for all $a \in A$ to produce a cover $\{U(a) \times V(a)\}_{a \in A}$ of $A \times B$ that is contained in $N$. As before, we can replace this by a finite subcover: associated to $a_1, \ldots , a_m$, say. Then set $U = \bigcap_{i=1}^{m} U(a_i)$ and $V = \bigcup_{i=1}^{m} V(b_i)$. The open set $U \times V$ satisfies the conditions of the lemma.

**Corollary 12.6** (The tube lemma). Let $X$ and $Y$ be topological spaces and suppose $X$ is compact. Let $y \in Y$. Suppose $N$ is an open neighbourhood of $X \times \{y\}$, then there exists an open $U \ni y$ such that $X \times U \subseteq N$.

### 13 Interlude on Normality

**Definition 13.1.** A topological space $X$ is **regular** if for all point $p \in X$ and all closed sets $C \subseteq X \setminus \{p\}$, there exists disjoint open sets $U \ni p$ and $V \ni C$ such that $U \cap V = \emptyset$.

**Remark 13.2.** There exist Hausdorff spaces that are not regular, and there exist regular spaces that are not Hausdorff (e.g., the indiscrete topology). A space that is both Hausdorff ($T_2$) and regular is called a **$T_3$-space**.

**Definition 13.3.** A topological space $X$ is **normal** if, for all disjoint closed subsets $C_1, C_2$, there exists open subsets $U_1 \ni C_1$ and $U_2 \ni C_2$ such that $U_1 \cap U_2 = \emptyset$.

**Remark 13.4.** A Hausdorff normal space is a **$T_4$-space**. Clearly, a $T_4$-space is a $T_3$-space.

**Example 13.5.** It is not easy to dream up a space that is $T_3$ but not $T_4$, but they exist. For instance, if we take $\mathbb{R}$, but give it the right-half-open interval topology, and then form $\mathbb{R} \times \mathbb{R}$, the resulting space, the “half-open square topology” is $T_3$ but not $T_4$. I don’t know a way to prove this that doesn’t involve the Baire category theorem.

**Proposition 13.6.** A compact Hausdorff space $X$ is a $T_4$ space.

**Proof.** In a time-honoured tradition, we prove that $X$ is $T_3$. The proof that it’s $T_4$ is the same argument again.

Let $p$ be a point and $C$ be a closed subset disjoint from $p$. Since $X$ is compact, $C$ is compact. Since $X$ is Hausdorff, for each $c \in C$ we can find disjoint $U_c \ni p$ and $V_c \ni c$ such that $U_c \cap V_c = \emptyset$. Finitely many $V_c$ suffice to cover $C$: say $V(p) = V_{c_1} \cup \cdots \cup V_{c_n} \ni C$. Then $U(p) = \bigcap_{i=1}^{n} U_{c_i}$ is an open set disjoint from $V(p)$ and $p \in U(p)$.

If $C_1$ and $C_2$ are disjoint closed sets, for each $p \in C_1$ we can find disjoint open $U(p) \ni p$ and $V(p) \ni C_2$. Finitely many of the $U(p)$ suffice to cover $C_1$ and we use the same finite-intersection idea again to prove $X$ is normal. 

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CHAPTER 3. COMPACTNESS

14 Compactifications

This proposition probably should have come earlier in the development:

**Proposition 14.1.** If $X$ is a space and $C_1$ and $C_2$ are two compact subsets, then $C_1 \cup C_2$ is compact.

**Proof.** A cover of $C_1 \cup C_2$ contains a finite subset covering $C_1$ and a finite subset covering $C_2$.  

**Notation 14.2.** We’ve used this idea before, but let’s formalize it: an embedding $i : X \to Y$ is a continuous function such that $i$ induces a homeomorphism $X \to i(X)$.

**Definition 14.3.** Let $X$ be a topological space. A compactification of $X$ is an embedding $i : X \to \hat{X}$ where $\hat{X}$ is a compact space and where the image of $i$ is dense in $\hat{X}$.

**Remark 14.4.** If $X$ is already compact, then the only compactification is $X$ itself, up to homeomorphism. On the other hand, if $X$ is not compact, there are likely to be many inequivalent compactifications.

**Construction 14.5.** Let $X$ be a topological space. Let the one-point compactification $X \cup \{1\}$ be the set consisting of $X$ itself and a new point “at infinity”. The topology on $X \cup \{1\}$ is defined as follows:

- open subsets of $X$ are open in $X \cup \{1\}$.
- A subset $U \ni \{1\}$ is open if $X \setminus (U \cap X)$ is a closed compact subset of $X$.

We note that this is actually a topology. Verifying this is routine.

**Remark 14.6.** In spite of the name, this is not a compactification if $X$ is already compact. In that case, $X \cup \{\infty\} = X$.

**Remark 14.7.** If $X$ is Hausdorff, then all compact subsets of $X$ are closed, so the “closed” in “closed compact” is redundant. This is the most commonly-used case of the construction.

**Proposition 14.8.** The one-point compactification of a non-compact space is a compactification—strictly, the obvious inclusion $i : X \to X \cup \{1\}$ is.

**Proof.** First we observe that $i : X \to X \cup \{\infty\}$ is an embedding. It is clearly injective and easy to verify that it is continuous. In fact, it is an open map so it follows that it is an embedding.

Next we prove that $X \cup \{\infty\}$ is compact. Suppose $\{U_i\}_{i \in I}$ is a cover. Then at least one $U_1 \ni \{1\}$, so that $X \setminus \{U_1\}$ is compact. Therefore finitely many of the other $U_i$ suffice to cover $X \setminus U_1$.

Finally, we verify that $X \subseteq X \cup \{\infty\}$ is dense. This requires us to know that $X$ is not compact itself, so that $\{\infty\}$ is not an open set. Therefore $X$ is not closed, so its closure must be $X \cup \{1\}$.  

**Example 14.9.** We have already seen a one-point compactification: $\mathbb{N} \cup \{\infty\}$ was used to define convergence of a sequence.

**Example 14.10.** Here is another, more troubling, example. Take $\mathbb{Q}$ with the usual topology and form the one-point compactification $\mathbb{Q} \cup \{\infty\}$. The open neighbourhoods of $\infty$ are the complements of compact sets. We claim that no compact set $K$ contains the intersection of $\mathbb{Q}$ with an open interval $I$. If it did, we could find a sequence in $I \cap \mathbb{Q}$ converging to an irrational number, and this sequence could have no convergent subsequence in $\mathbb{Q}$.

Therefore, for any $q \in \mathbb{Q}$, every open neighbourhood $(a, b) \cap \mathbb{Q}$ meets every open neighbourhood of $\infty$. We deduce that $\mathbb{Q} \cap \{\infty\}$ is compact, but it is not Hausdorff.
Nonetheless, something funny happens in this space about the limits of convergent sequences—they are still unique even though the space is not Hausdorff. The technical term is that the space is weak Hausdorff: this will be defined in the homework.

Now we devote ourselves to finding conditions that ensure that the one-point compactification of a Hausdorff space is again Hausdorff.

**Definition 14.11.** We will say a space $X$ is **locally compact** if every point $x \in X$ is contained in an open set $U$ that is itself contained in a compact set $K$.

**Proposition 14.12.** If $X$ is a Hausdorff space then the following are equivalent:

1. $X$ is locally compact,
2. every point $x \in X$ has an open neighbourhood $U$ such that $\bar{U}$ is compact,
3. every point $x \in X$ has a local base consisting of open sets $U$ such that $\bar{U}$ is compact.

**Proof.** It is trivial that (iii) implies (ii) which implies (i). Therefore it suffices to show that (i) implies (iii). Let $x \in X$ be a point and choose $U \subseteq K$ such that $x \in U$ and $U \subseteq K$ where $U$ is open and $K$ is compact. Let $\{V_i\}_{i \in I}$ be any local base at $x$. Then $\{U \cap V_i\}_{i \in I}$ is another local base at $x$ and, furthermore, each of these sets is contained in $K$, which is closed—due to the Hausdorff property. Therefore their closures are closed and contained in $K$. Consequently, their closures are compact.

**Proposition 14.13.** Let $X$ be a locally compact Hausdorff space. Then the one-point compactification $X \cup \{\infty\}$ is Hausdorff.

**Proof.** Let $x \neq y$ be two points in $X \cup \{\infty\}$. These two points have disjoint open neighbourhoods in $X$ if they both lie in $X$. Therefore the only case we have to check is when $y = \{\infty\}$. We can find an open $U \ni x$ such that the closure in $X$, denoted $\bar{U}$ is compact, and so $X \cup \{\infty\} \setminus \bar{U}$ is an open neighbourhood of $\{\infty\}$ disjoint from $U$.

**Lemma 14.14.** Let $i : X \to Y$ be any Hausdorff compactification of a locally compact Hausdorff space. Then $i$ is an open map.

**Proof.** Since $i : X \to i(X)$ is a homeomorphism, it suffices to prove that $i(X)$ is open in $Y$. Let $x \in X$. There exists some open $U \ni x$ and a compact $U \subseteq K$ in $X$. We know that $i(K)$ is compact, and hence closed, in $Y$. Since $i(X)$ is homeomorphic to $X$, there exists an open $W \ni i(x)$ such that $W \cap i(K) = i(U)$. Now consider $W \setminus i(K)$. This is an open set in $Y$ and $W \cap i(K) \subseteq i(K)$ implies that it is disjoint from $i(X)$. Since $i(X)$ is dense in $Y$, this means that it must be empty, so $W \subseteq i(K) \subseteq i(X)$. It follows $i(X)$ is open.

For any open $U \subseteq X$, the set $i(U) = V \cap i(X)$ for some open $V \subseteq Y$, since $i$ is a homeomorphism onto its image. Since $i(X)$ is open, the map $i : X \to Y$ is open too.

**Proposition 14.15.** Let $X$ be a locally compact, Hausdorff, non-compact space and suppose $i : X \to Y$ is a compactification where $Y$ is Hausdorff. Then the map $f : Y \to X \cup \{\infty\}$ given by

$$f(y) = \begin{cases} y & \text{if } y \in X \\ \infty & \text{otherwise} \end{cases}$$

is continuous.
This means that \( X \cup \{\infty\} \) is final among all Hausdorff compactifications of \( X \).

**Proof.** Let \( U \subseteq X \cup \{\infty\} \) be an open set. We want to show that \( f^{-1}(U) \) is open. There are two cases to consider.

First consider the case where \( \infty \not\in U \). In this case, the argument is clear enough: \( U \subseteq X \) is open, so \( i(U) = f^{-1}(U) \) is open by Lemma 14.14.

Second, consider the case when \( \infty \in U \). Then \( X \cup \{\infty\} \setminus U \) is a compact subset \( K \) of \( X \), and \( i(K) = f^{-1}(X \cup \{\infty\} \setminus U) \) is closed in \( Y \). The result follows.

**Corollary 14.16.** Let \( X \) be a locally compact, Hausdorff but not compact space, and let \( i : X \to Y \) be a Hausdorff compactification in which \( Y \setminus i(X) \) consists of one point. Then \( i \) is a one-point compactification.

**Proof.** There exists a continuous bijection \( f : Y \to X \cup \{\infty\} \) in which the source is compact and the target is Hausdorff.

**Example 14.17.** Using stereographic projection, we can prove that \( S^n \) is a one-point compactification of \( \mathbb{R}^n \). This is on the homework.