Chapter 2

Closure and sequence methods

5 Closure

Definition 5.1. Let $X$ be a topological space and let $A$ be a subset of $X$. The closure of $A$ in $X$, written $\overline{A}$, is the intersection of all closed sets that contain $A$.

Proposition 5.2. The closure operator on subsets of a topological space $X$ has the following properties:

1. $\overline{A}$ is closed.
2. $A \subseteq \overline{A}$, with equality if and only if $A$ is closed.
3. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$
4. $\overline{\overline{A}} = \overline{A}$

Proof. 1. Since the intersection of closed subsets is closed, this is immediate.
2. This is the case because $\overline{A}$ is the intersection of closed sets, all containing $A$.
3. Any closed set containing $B$ is a closed set containing $A$. Therefore $\overline{A}$ is the intersection of a family of sets that contains all closed sets containing $B$. The result follows.
4. This is immediate, since $\overline{A}$ is closed.

Proposition 5.3. With notation as before, $x \in \overline{A}$ if and only if, for every open neighbourhood $U \ni x$, the set $U \cap A \neq \emptyset$.

Proof. Consider the statement “every open neighbourhood $U$ of $x$ satisfies $U \cap A \neq \emptyset$”. This is logically equivalent to “for every open set $U$ such that $U \cap A = \emptyset$, the element $x$ is not in $U$”, which is equivalent to “for every closed set $C$ such that $A \subseteq C$, the element $x$ is in $C$” which is equivalent to “$x \in \overline{A}$”.

Remark 5.4. Given a sequence of inclusions of sets $Z \subseteq Y \subseteq X$, where $X$ is a topological space, it may be the case that the closure of $Z$ in the subspace topology on $Y$ is different from the closure of $Z$ in the topology on $X$. On the other hand, if $Y \subseteq X$ is closed, then the two notions of closure coincide.
6 Interior and boundary

Definition 6.1. Suppose $A \subseteq X$. Let $A^\circ$, the interior of $A$, denote the union of all open $U \subseteq A$.

This concept is dual to that of closure. It is immediate that $A^\circ \subseteq A$, with equality if and only if $A$ is open.

Definition 6.2. Suppose $A \subseteq X$. Let $\partial A$, the boundary of $A$, denote $\overline{A} - A^\circ$.

Proposition 6.3. With notation as above, a point $x \in X$ lies in $\partial A$ if and only if every neighbourhood $U \ni x$ satisfies $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$.

Proof. The proof of this is an exercise. □

Proposition 6.4. Let $X$ be a topological space and let $A$ be a subspace. Then there is a division of $X$ into three disjoint subsets: $A^\circ$, $\partial A$ and $(X - A)^\circ$. Moreover $\overline{A} = A^\circ \cup \partial A$.

Proof. The division of $X$ into three parts is really as follows: let $x \in X$ be a point. Then exactly one of the following three cases must obtain:

1. There is some open $U \ni x$ such that $U \subseteq A$. In this case, $x \in A^\circ$.

2. There is some open $U \ni x$ such that $U \subseteq X - A$. In this case, $x \in (X - A)^\circ$.

3. For every open $U \ni x$, both $U \cap A$ and $U \cap (X - A)$ are not empty. In this case, $x \in \partial X$.

The closure of $A$ consists of those $x$ for which every open neighbourhood meets $A$, by Proposition 5.3. Therefore $\overline{A}$ is the complement of $(X - A)^\circ$. The result follows. □

Corollary 6.5. $\partial A = \partial (X - A)$.

7 Density

Definition 7.1. We say a subset $A \subseteq X$ is dense if $\overline{A} = X$. Dually, we say $A$ is sparse or nowhere dense if $(\overline{A})^\circ = \emptyset$.

Remark 7.2. The set $A$ is dense in $X$ if $X$ is the only closed set containing $A$. The contrapositive is that the only open set $UX \setminus A$ is the empty set. Therefore a set $A$ is dense if and only if $U \cap A$ is nonempty whenever $U$ is a nonempty open subset.

Example 7.3. The subset $\mathbb{Q} \subseteq \mathbb{R}$ is dense. On the other hand, the Cantor set $C \subseteq [0,1]$ consisting of all the numbers without a 2 in their (nonterminating) ternary representation is nowhere dense.

Proposition 7.4. Let $X$ be a topological space, let $AX$ be a dense subset, and let $Y$ be a Hausdorff topological space. Suppose $f, g : X \to Y$ are two continuous functions such that $f(a) = g(a)$ for all $a \in A$. Then $f = g$.

In proving this, we use some lemmas that are useful in their own right. You can prove the result much more directly, but the following are all likely to be useful.

Definition 7.5. Let $X$ be a topological space. Write $\Delta$ for the diagonal subset of $X \times X$ consisting of points $(x,x)$. 
Remark 7.6. There is an obvious function $\Delta : X \to X \times X$, the image of which is the diagonal. This function is continuous (why?) and bijective. It is actually a homeomorphism: consider an open set $U \subseteq X$, and the projection $p_1 : X \times X \to X$. Then $\Delta \cap p_1^{-1}(U) = \Delta(U)$.

**Lemma 7.7.** Let $Y$ be a topological space. Then $Y$ is Hausdorff if and only if the diagonal subset $\Delta(Y) \subseteq Y \times Y$ consisting of points $(y, y)$ is a closed subset.

**Proof.** Suppose $Y$ is Hausdorff. Let $(x, y)$ be a non-diagonal point. Then there exists open $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$ and $U \times V$ therefore gives a subset of $Y \times Y$ containing $(x, y)$ and disjoint from $\Delta$. It follows that $Y \times Y - \Delta$ is open.

Suppose $\Delta$ is closed. Let $(x, y)$ be a point not in the diagonal. Then there exists some open sets $U \ni x$ and $V \ni y$ such that $(U \times Y) \cap (Y \times V)$ does not meet the diagonal. But this implies that $U \cap V = \emptyset$, as required.

**Lemma 7.8.** Let $f, g : X \to Y$ be two continuous functions, and suppose $Y$ is Hausdorff. The set of points $x \in X$ such that $f(x) = g(x)$ is closed in $X$.

**Proof.** We can identify $X$ with $\Delta \subseteq X \times X$. The function $(f \times g) : X \times X \to Y \times Y$ is continuous (why?) and the inverse image $(f \times g)^{-1}(\Delta)$ is therefore closed in $X \times X$. Therefore $(f \times g)^{-1}(\Delta) \cap \Delta$ is closed in $\Delta$, but this consists of precisely those $x$ such that $f(x) = g(x)$.

Of course, you can do this much more directly if you like.

**Proof of Proposition.** The set of points where $f(x) = g(x)$ is closed, and contains the dense subset $A$. Therefore it is all of $X$. \qed

8 Sequences

I assume you know what a sequence $(x_1, x_2, \ldots)$ in a topological space $X$ means. We’ll denote sequences by $x_n$, or, when it is necessary to specify the indexing variable, $(x_n)_n$.

**Definition 8.1.** A sequence $(x_1, x_2, \ldots)$ in $X$ converges to $x \in X$ if, for all open $U \ni x$, there exists some $N \in \mathbb{N}$ such that $x_i \in U$ for all $i > N$.

In this formulation, it is clear that the notion of convergence given here is a generalization of the notion you are familiar with from analysis.

Here is a different way of conceptualizing convergence.

**Notation 8.2.** Unless we say otherwise, the set $\mathbb{N} \cup \{\infty\}$ will be given a topology where $\{n\}$ is open for all $n \in \mathbb{N}$, and a set $U \ni \infty$ is open if and only if it contains some tail: $\{n, n+1, n+2, \ldots\}$.

The induced subspace topology on $\mathbb{N}$ is discrete.

**Remark 8.3.** We can conceptualize a sequence $x_n$ as a function $x : \mathbb{N} \to X$. Since $\mathbb{N}$ carries the discrete topology, all such functions are continuous.

Suppose this sequence $x : \mathbb{N} \to X$ extends to a continuous function $\hat{x} : \mathbb{N} \cup \{\infty\} \to X$. Then this is equivalent to saying $x_n \to \hat{x}(\infty)$.

**Remark 8.4.** The subspace $\mathbb{N}$ is dense in $\mathbb{N} \cup \{\infty\}$.

In this formulation, the following is an immediate consequence of the fact that composites of continuous functions are continuous.
Proposition 8.5. Let \( x_n \) be a sequence in \( X \) and suppose \( x_n \) converges to \( x \). Let \( f : X \to Y \) be a continuous function. Then \( f(x_n) \) converges to \( f(x) \) in \( Y \).

Remark 8.6. Similar tricks with composite functions \( \mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\} \) can be used to show that if \( x_n \to x \) and if \( y_n \) is a subsequence of \( x_n \), then \( y_n \to x \). You can also do this directly. You probably get enough of this sort of thing in analysis lectures.

We know from analysis lectures that the limit of a convergent sequence is unique in a metric space. Unfortunately, this does not generalize to non-Hausdorff spaces.

Example 8.7. Consider an infinite set \( X \) with the cofinite topology. Let \( x_n \) be a sequence in \( X \) in which \( x_i \neq x_j \) for all \( i \neq j \). Such a sequence exists because there exists an injective map \( \mathbb{N} \to X \).

Let \( y \in X \). Consider any open \( U \ni y \). The set \( X \setminus U \) consists of only finitely many elements, and therefore only finitely many elements of \( (x_n) \) lie outside \( U \). In particular, some tail of the sequence \( (x_n) \) lies entirely inside \( U \), and so \( x_n \to y \). But \( y \) was arbitrary.

This shows that even in a \( T_1 \) topological space, limits of sequences may not be unique.

Proposition 8.8. Let \( X \) be a Hausdorff topological space and let \( x_n \) be a sequence in \( X \). Suppose \( x_n \to y \) and \( x_n \to z \). Then \( y = z \).

Proof. The two limits can be encoded as continuous functions

\[
\begin{array}{ccc}
\mathbb{N} & \xrightarrow{\hat{y}} & \mathbb{N} \\
& & \\
& \downarrow & \\
\mathbb{N} \cup \{\infty\} & \xrightarrow{\hat{z}} & X
\end{array}
\]

Here \( \hat{y}(0) = y \) and \( \hat{z}(0) = z \), whereas \( \hat{y}|_{\mathbb{N}} = \hat{z}|_{\mathbb{N}} \). Since \( \mathbb{N} \) is dense in \( \mathbb{N} \cup \{\infty\} \), and \( X \) is Hausdorff, it follows that \( \hat{y} = \hat{z} \).

Of course, one can prove this in a lower-level way, by working directly with points and sets. You have surely all done this in analysis courses.

Our function-based approach to sequence convergence gives us a slick proof of the following:

Proposition 8.9. Let \( \{X_i\}_{i \in I} \) be a family of topological spaces, let \( X \) denote the product. Let \( (x_i) \) be a sequence in \( X \). Let \( y \in X \) be an element. Then \( (x_i) \to y \) if and only if \( \pi_i(x_i) \to \pi_i(y) \) for all \( i \).

That is, a sequence in a product space converges to \( y \) if and only if all the projections converge to the appropriate projections of \( y \).

In a metric space, the closed sets admit a characterization in terms of limits of sequences. This carries over to all first-countable spaces.

Definition 8.10. Let \( X \) be a topological space and \( A \) a subspace. Say that \( A \) is sequentially closed if it has the following property: if \( (a_n)_n \) is a sequence in \( A \) that converges to \( x \in X \), then \( x \in A \).

Proposition 8.11. Let \( X \) be a topological space and \( A \) a closed subspace. Then \( A \) is sequentially closed.
Proof. Consider the commutative diagram of continuous maps

\[
\begin{array}{ccc}
\mathbb{N} & \longrightarrow & \mathbb{N} \cup \{\infty\} \\
\downarrow a & & \downarrow 1 \\
A & \longrightarrow & X
\end{array}
\]

which implements \((a_n)_n \to x\). If \(A\) is closed in \(X\), then \(\hat{x}^{-1}(A)\) is closed in \(\mathbb{N}\). Since \(\hat{x}^{-1}(A) \supseteq \mathbb{N} = \mathbb{N} \cup \{\infty\}\).

**Proposition 8.12.** Suppose \(X\) is a first-countable topological space and \(A\) is a sequentially closed subspace. Then \(A\) is closed.

Proof. Let \(x \in \mathring{A}\) be a point in the closure of \(A\). We want to show \(x \in A\). Let \(U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots\) be a sequence of open neighbourhoods of \(x\) that form a local base for \(x\) in \(X\). By Proposition 5.3, we can find elements \(a_i \in U_i \cap A\). Define a function \(\hat{x} : \mathbb{N} \cup \{\infty\} \to X\) by \(n \mapsto a_n\) and \(\infty \mapsto x\). This function is continuous: If \(V \subseteq X\) is open but does not contain \(x\), then \(\hat{x}^{-1}(V)\) does not contain \(\infty\), so is open; whereas if \(V \ni x\), then there exists some \(U_j \subseteq V\), and so \(\hat{x}^{-1}(V) \supseteq \hat{x}^{-1}(U_j) = \{i, i+1, \ldots\} \cup \{\infty\}\), so \(\hat{x}^{-1}(V)\) is open in \(\mathbb{N} \cup \{\infty\}\).

Therefore \(x\) is the limit of a sequence in \(A\), and so \(x \in A\), as required.

**Corollary 8.13.** Let \(X\) and \(Y\) be topological spaces where \(X\) is first countable. Suppose \(f : X \to Y\) is a function with the property that \(x_n \to x\) in \(X\) implies \(f(x_n) \to f(x)\) in \(Y\). Then \(f\) is continuous.

Proof. Let \(A \subseteq Y\) be a closed set. We will show \(f^{-1}(A)\) is closed. If \(A\) is empty, there is nothing to show, so assume it is not. To show \(f^{-1}(A)\) is closed, it is sufficient to show it is sequentially closed. Let \(x_n\) in \(f^{-1}(A)\) be a sequence converging to \(x \in X\). Then \(f(x_n)\) is a sequence in \(A\) converging to \(f(x)\), and since \(A\) is closed, \(f(x) \in A\), which implies \(x \in f^{-1}(A)\).

\[\square\]