## Contents

1 Topological spaces
   1.1 Basic definitions ............................................ 7
   1.2 Separation axioms ........................................... 8
   1.3 Continuous functions ....................................... 10
   1.4 Generating topologies ...................................... 12
   1.5 Induced topologies ........................................... 14
   1.6 Coinduced topologies ....................................... 16

2 Closure and sequence methods
   2.1 Closure ......................................................... 19
   2.2 Interior and boundary ....................................... 20
   2.3 Density ........................................................ 21
   2.4 Sequences ...................................................... 22
   2.5 Completions of Metric Spaces ............................... 24

3 Compactness
   3.1 Elementary Theory .......................................... 27
   3.2 The Tube Lemma .............................................. 28
   3.3 Compactness in Metric Spaces ............................... 30
   3.4 Tychanoff’s Theorem ......................................... 33
   3.5 Compactifications ............................................ 34
   3.6 Compactly generated topologies ............................ 38

4 Connectedness
   4.1 Connectedness ................................................. 41
   4.2 Path-connectedness .......................................... 43
## 5 Homotopy
5.1 The homotopy category .................................. 45
5.2 The pointed homotopy category and homotopy groups .... 47
5.3 Contractible spaces ........................................ 50

## 6 The fundamental groupoid and the fundamental group
6.1 The fundamental groupoid .................................. 53
6.2 More about groupoids ....................................... 55
6.3 Fundamental groupoids of spaces again .................... 57
6.4 Basepoints and the fundamental group .................... 57
6.5 The van Kampen theorem .................................. 58
6.6 Examples .................................................. 63

## 7 Covering spaces
7.1 Covering Spaces ............................................ 71
7.2 Fundamental group action .................................. 75
7.3 Fundamental groups of quotients ......................... 77
7.4 Lifting and covering spaces ................................ 81
7.5 The classification of covering spaces .................... 82
7.6 Deck transformations ....................................... 87

## A Quotients and group actions
A.1 More about quotients ...................................... 89
A.2 Topological Groups ........................................ 91
A.3 Group actions ............................................. 92
A.4 Quotients by group actions ................................ 93

## B The Compact–Open Topology
B.1 Definition .................................................. 95
B.2 Currying and Uncurrying .................................. 96
B.3 The pointed case ........................................... 98

## C Category Theory
C.1 Categories, Functors and Natural Transformations ....... 99
C.2 Functors and Natural Transformations .................... 102
C.3 Adjoint Functors ......................................... 105
C.4 Diagrams, Limits and Colimits ............................ 107
D \hspace{1em} p-Norms \hspace{1em} 113
  D.1 The $p$ norms on $\mathbb{R}^n$ \hspace{1em} 113
  D.2 Norms and metrics \hspace{1em} 116
  D.3 The $p$-norms and product metrics \hspace{1em} 117
  D.4 The $p$-norms on sequence spaces \hspace{1em} 118
    D.4.1 Completeness \hspace{1em} 123
  D.5 The $p$-norms for functions \hspace{1em} 123

E \hspace{1em} Urysohn's Lemma and the Tietze Extension Theorem \hspace{1em} 125
  E.1 Urysohn's Lemma \hspace{1em} 125
  E.2 Tietze Extension \hspace{1em} 126
Chapter 1

Topological spaces

1.1 Basic definitions

Definition 1.1. Let $X$ be a set and $\tau \subseteq \mathcal{P}X$ a subset of the power set of $X$, i.e., $\tau$ is a set of subsets of $X$. We say $(X, \tau)$ is a topological space and $\tau$ is a topology on $X$ if the following axioms are satisfied:

1. $\emptyset \in \tau$ and $X \in \tau$.

2. If $\{U_i\}_{i \in I} \subset \tau$, then $\bigcup_{i \in I} U_i \in \tau$. That is, $\tau$ is closed under taking arbitrary unions.

3. If $\{U_i\}_{i=1}^n \subset \tau$ is a finite subset, then $\bigcap_{i=1}^n U_i \in \tau$. That is, $\tau$ is closed under taking finite intersections.

Notation 1.2. The sets $U_i \in \tau$ are called open sets for the topology $\tau$. A set $C = X \setminus U$ where $U$ is open is called closed. The closed sets satisfy a dual set of axioms:

1. $\emptyset$ and $X$ are both closed.

2. An arbitrary intersection of closed sets is closed.

3. A finite union of closed sets is closed.

Definition 1.3. Let $(X, d)$ be a metric space. That is $d : X \times X \to [0, \infty)$ is a metric satisfying the usual metric axioms:

1. $d(x, y) = d(y, x)$

2. $d(x, z) \leq d(x, y) + d(y, z)$
3. \( d(x, y) = 0 \) if and only if \( x = y \)

Given a point \( x \in X \) and \( r > 0 \), we define the open ball around \( x \) of radius \( r \), denoted \( B(x, r) \) to be \( \{ y \in X \mid d(x, y) < r \} \).

**Example 1.4.** Then we can define an associated metric topology \( \tau_d \) on \( X \) as follows: A set \( U \) is open in the metric topology if: for any point \( u \in U \), there exists some \( \epsilon > 0 \), possibly depending on \( u \), such that the ball \( B(u; \epsilon) \subseteq U \).

It is an easy exercise to prove that this really is a topology as defined above.

**Proposition 1.5.** Let \( B(x, r) \) be an open ball in a metric space. Then \( B(x, r) \) is an open set.

**Proof.** Let \( y \in B(x, r) \) be a point. We know that \( d(y, x) < r \). Write \( d(y, x) = r - \epsilon \) for some \( \epsilon \). We claim that \( B(y, \epsilon) \subseteq B(x, r) \). Suppose \( z \in B(y, \epsilon) \), then

\[
d(x, z) \leq d(x, y) + d(y, z) < r - \epsilon + \epsilon = r
\]

so that \( z \in B(x, r) \). \( \square \)

1.2 Separation axioms

**Notation 1.6.** If \((X, \tau)\) is a topological space and \( x \in X \) is a point, then an open set \( U \ni x \) will be called an (open) neighbourhood of \( x \). A set \( Y \ni x \) such that there exists an open set \( Y \supseteq U \ni x \) is sometimes also called a neighbourhood. We will flag this usage up if we have to use it later.

**Definition 1.7.** A topological space \((X, \tau)\) is T1 if all singleton subsets \( \{x\} \subseteq X \) are closed.

**Remark 1.8.** This axiom is equivalent to stating that for any \( x \in X \) and \( y \in X \setminus \{x\} \), there exists some open set \( U \ni y \) but \( U \notni x \). The proof of this follows from a basic argument we will use over and over.

**Proposition 1.9.** A subset \( U \) in a topological space \((X, \tau)\) is open if and only if for all \( u \in U \), there is some open neighbourhood \( V \ni u \) such that \( U \supseteq V \).

**Proof.** In one direction, if \( U \) is open, we can take \( U = V \).

In the other, if \( U \) contains open neighbourhoods \( V_u \) around each of its points, then

\[
U = \bigcup_{u \in U} V_u
\]

is open. \( \square \)
Therefore, if \((X, \tau)\) has the property that for every pair \(u \in X\) and \(v \in X\) with \(u \neq v\), there exists an open neighbourhood \(W_v \ni v\) but \(W_v \not\ni u\), then we can write

\[
X \setminus \{u\} = \bigcup_{v \in X \setminus \{u\}} W_v
\]

which is open.

**Definition 1.10.** A topological space is \(T_2\) or *Hausdorff* if, for any \(x \neq y \in X\), there exist open sets \(U \ni x\) and \(V \ni y\) such that \(U \cap V = \emptyset\).

**Remark 1.11.** This is clearly a stronger condition than the \(T_1\) property. That is, points in Hausdorff spaces are closed.

**Proposition 1.12.** Every metric topology is Hausdorff.

**Proof.** Let \(x \neq y\) be two points in a metric space. We can write \(d(x, y) = 2\varepsilon\) where \(\varepsilon > 0\). Then \(B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset\), by using the triangle inequality. \qed

**Example 1.13.** Let \(X\) be set. Give \(X\) the *indiscrete topology* where the open sets consist only of \(\emptyset\) and \(X\). If \(X\) contains at least 2 elements, then this topology is not \(T_1\) and therefore it is not Hausdorff and therefore it is not metric.

**Example 1.14.** Let \(X\) be a set and define the *cofinite topology* on \(X\) as follows: \(U \subseteq X\) is open if \(X \setminus U\) is finite or if \(U = \emptyset\). This always yields a topology. If \(x \neq y\) are two elements in \(X\), then \(X \setminus \{y\}\) is an open set containing \(x\) but not containing \(y\), so the topology is \(T_1\). On the other hand, if \(X\) is infinite, and if \(U \ni x\) and \(V \ni y\) are open neighbourhoods, then \((X \setminus U) \cup (X \setminus V)\) is a finite set. In particular, it is not all of \(X\). Any \(z\) in the complement lies in both \(U\) and \(V\), and so \(U \cap V \neq \emptyset\). Therefore, this topology is not Hausdorff (and in particular, is not metric).

**Example 1.15.** A useful, if uncomplicated, topology is the *discrete topology*. Here every set is open, and consequently every set is closed. It is a metric topology, being induced by the discrete metric \(d(x, y) = 1\) if \(x \neq y\) for instance.

**Definition 1.16.** A topological space \(X\) is *regular* if for all point \(p \in X\) and all closed sets \(C \subseteq X \setminus \{p\}\), there exists disjoint open sets \(U \ni p\) and \(V \ni C\) such that \(U \cap V = \emptyset\).

**Remark 1.17.** There exist Hausdorff spaces that are not regular, and there exist regular spaces that are not Hausdorff (e.g., the indiscrete topology). A space that is both Hausdorff \((T_2)\) and regular is called a \(T_3\)-space.

**Definition 1.18.** A topological space \(X\) is *normal* if, for all disjoint closed subsets \(C_1, C_2\), there exists open subsets \(U_1 \ni C_1\) and \(U_2 \ni C_2\) such that \(U_1 \cap U_2 = \emptyset\).
1.3 Continuous functions

Remark 1.19. A Hausdorff normal space is a $T_4$-space. Clearly, a $T_4$-space is a $T_3$-space.

Example 1.20. It is not easy to dream up a space that is $T_3$ but not $T_4$, but they exist. For instance, if we take $\mathbb{R}$, but give it the right-half-open interval topology from Example 1.46, and then form $\mathbb{R} \times \mathbb{R}$, the resulting space, the “half-open square topology” is $T_3$ but not $T_4$. I don’t know a way to prove this that doesn’t involve the Baire category theorem.

1.3 Continuous functions

The purpose of topological spaces is to define continuity. First we recall the metric-space definition of continuity.

Definition 1.21. Let $(X_1, d_1)$ and $(X_2, d_2)$ be two metric spaces, and let $f : X_1 \to X_2$ be a function. We say that $f$ is continuous if, for all $x \in X_1$ and all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon.$$  

Definition 1.22. Let $(X_1, \tau_1)$ and $(X_2, \tau_2)$ be topological spaces and let $f : X_1 \to X_2$ be a function. We say $f$ is continuous if, for all open sets $U \subseteq X_2$, the preimage set $f^{-1}(U) = \{x \in X_1 \mid f(x) \in U\}$ is open in $X_1$.

Proposition 1.23. Let $(X_1, d_1)$ and $(X_2, d_2)$ be metric spaces and $f : X_1 \to X_2$ be a function. Then $f$ is continuous in the metric-space sense if and only if it is continuous in the topological sense.

Proof. Suppose $f$ is metric-continuous. Let $U \subseteq X_2$ be an open set. If $U$ is empty, then there is nothing to check. Suppose $x \in f^{-1}(U)$, so that $f(x) \in U$. Since $U$ is open, there is some $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq U$. Then choose an associated $\delta$, so that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$. This last condition is equivalent to saying that $f(y) \in B(f(x), \epsilon)$, so that $y \in f^{-1}(U)$. We have shown that $B(x, \delta) \subseteq f^{-1}(U)$. Since $f^{-1}(U)$ contains an open ball around each of its points, it is an open set.

Conversely, suppose $f$ is topologically continuous. Let $x \in X$ and $\epsilon > 0$. The set $B(f(x), \epsilon)$ is open in $X_2$, and therefore $f^{-1}(B(f(x), \epsilon))$ must be open in $X$. That implies that there exists some $\delta$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$, which is a restatement of the $\epsilon$-$\delta$-continuity condition. □

Remark 1.24. We will assume therefore that functions from calculus classes etc. that you might reasonably have proved to be continuous are continuous.
Proposition 1.25. Let \( f : X \to Y \) and \( g : Y \to Z \) be continuous functions between topological spaces. Then \( g \circ f \) is continuous.

Proof. Once you observe that \((g \circ f)^{-1}(U) = g^{-1}(f^{-1}(U))\), this is immediate. \( \square \)

Proposition 1.26. Let \( f : X \to Y \) be a function between topological spaces. Then \( f \) is continuous if and only if \( f^{-1} \) takes closed sets to closed sets.

Proof. Once you observe that \( f^{-1}(Y \setminus Z) = X \setminus f^{-1}(Z) \), this is immediate. \( \square \)

Proposition 1.27. Let \( X \) be a topological space with the discrete topology, and \( Y \) a topological space. Then every function \( f : X \to Y \) is continuous. Conversely, let \( Z \) be a topological space with the indiscrete topology. Then every function \( g : Y \to Z \) is continuous.

Definition 1.28. A function \( f : X \to Y \) is open (resp. closed) if \( f(U) \) is open (resp. closed) whenever \( U \) is open (resp. closed) in \( X \).

Both open and closed conditions on functions do arise in topology, but neither is comparable in importance to continuity.

Definition 1.29. A continuous function \( f : X \to Y \) with a continuous inverse \( f^{-1} : Y \to X \) is called a homeomorphism. This is the topological version of an isomorphism.

The proof of the following statement is elementary.

Lemma 1.30. If \( f : X \to Y \) is a continuous bijective function between topological spaces, then the following are equivalent:

1. \( f \) is a homeomorphism;

2. \( f \) is open;

3. \( f \) is closed.

Example 1.31. Let \( X \) be a set with at least two elements. Let \((X, i)\) denote the space \( X \) with the indiscrete topology and \((X, d)\) the space with the discrete topology. Then \( \text{id} : (X, d) \to (X, i) \) is continuous but not open or closed, whereas \( \text{id} : (X, i) \to (X, d) \) is open and closed, but not continuous.

The first of these two maps is an example of a continuous bijective function that is not a homeomorphism.

Example 1.32. The function \( f : [0, 2\pi) \to S^1 \) given by \( f(\theta) = e^{i\theta} \) is a continuous and bijective function, but it is not a homeomorphism. For example, the set \([0, \pi) \subset [0, 2\pi)\) is an open subset, but \( f([0, \pi)) \) is not open in \( S^1 \).
1.4 Generating topologies

**Definition 1.33.** Let \((X, \tau)\) be a topological space and let \(x \in X\) be a point. We say that a collection \(\mathcal{B}_x\) of open neighbourhoods is a **local base** for \(\tau\) at \(x\) or a **system of neighbourhoods of \(x\)** if, for all open \(U \ni x\), there exists at least one \(B \in \mathcal{B}_x\) such that \(x \in B \subseteq U\).

**Example 1.34.** The most common example of this phenomenon is the system of balls in metric spaces. Let \(x \in X\) be a point in a metric space and let \((a_n)_n \to 0\) be a sequence converging to 0, for instance \((1, 1/2, 1/3, \ldots)\). Then the family of balls \(\{B(x, a_i)\}_{i=1}^{\infty}\) is a local base for the topology on \(X\) at \(x\).

Note that this family is **countable**, that is, it can be indexed by the natural numbers.

**Proposition 1.35.** Suppose \(X\) is a topological space equipped with local bases \(\mathcal{B}_x\) at each point \(x\). The following are equivalent:

1. The function \(f\) is open;
2. For all \(x \in X\) and all \(B \in \mathcal{B}_x\), the set \(f(B)\) contains an open neighbourhood of \(f(x)\).

**Proof.** Suppose \(f\) is open, then \(f(B)\) is open, and therefore \(f(B)\) is a neighbourhood of \(f(x)\). This establishes one direction.

In the other direction, suppose \(U\) is an open set of \(X\). We wish to show \(f(U)\) is open. Suppose \(y \in f(U)\), then there exists some \(x \in U\) such that \(f(x) = y\). We can find a neighbourhood \(B \ni x\) such that \(B \subseteq U\) and then \(f(B) \subseteq f(U)\), but \(f(B)\) contains a neighbourhood of \(f(x) = y\) by hypothesis. Therefore \(f(U)\) contains a neighbourhood of the arbitrarily-chosen point \(y\), so that \(f(U)\) is open. \(\square\)

**Definition 1.36.** Let \(X\) be a topological space. If each point \(x \in X\) admits a countable local base, then we say \(X\) is **first countable**.

Every metric space is first countable.

**Definition 1.37.** Let \((X, \tau)\) be a topological space and let \(\mathcal{B}\) be a set of subsets of \(X\). If \(\mathcal{B}\) has the property that it contains a local base for \(\tau\) at every point, then \(\mathcal{B}\) is a **base** (or **basis**) for the topology \(\tau\).

**Example 1.38.** The prototypical example of a base is the collection of all balls of all radii in a metric topology.

**Definition 1.39.** Let \(X\) be a topological space. If \(X\) has a countable base, then \(X\) is **second countable**.
Example 1.40. The spaces $\mathbb{R}^n$ with the usual (metric) topology are second countable. This is perhaps surprising, but one can take as a countable basis all balls $B(q; 1/n)$ where $n \in \mathbb{N}$ and where $q$ has only rational-number coefficients. We will revisit this point.

Note that a base for a topology is not a wholly arbitrary set of subsets. For instance, if $A, B \in \mathcal{B}$ are sets in a base, and if $x \in A \cap B$, then there exists some $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

Definition 1.41. Let $\mathcal{S}$ be a set of subsets of a set $X$. Let $\mathcal{B}$ denote the set of all finite intersections of sets in $\mathcal{S}$. We say $\mathcal{S}$ is a subbase for a topology $\tau$ if $\mathcal{S}$ consists of open sets of $\tau$ and if $\mathcal{B}$ is a base for this topology. That is, for any open set $U \in \tau$ and any $u \in U$, there exists a finite intersection $x \in S_1 \cap \cdots \cap S_n \subseteq U$.

Whereas the condition of being a base of a topology places a condition on a set, any collection of subsets can form a subbase.

Construction 1.42. Let $X$ be a set and let $\mathcal{S}$ be a set of subsets of $X$. Let $\mathcal{B}$ denote the set of finite intersections of sets in $\mathcal{S}$ and let $\tau$ denote the set of unions of all sets in $\mathcal{B}$. Then $\tau$ is a topology, and $\mathcal{S}$ is a subbase of $\tau$.

Notation 1.43. We say that $\tau$ is the topology generated by $\mathcal{S}$.

Remark 1.44. In this definition, we use the convention that the intersection of an empty set of subsets of $X$ is $X$ itself. In order to avoid using this convention, some might prefer to impose a condition on a subbase $\mathcal{S}$ that the union of all sets in the subbase is $X$ itself.

Subbases can be used to detect continuity of functions:

Proposition 1.45. Let $f : X \to Y$ be a function between topological spaces. Let $\mathcal{S}$ generate the topology on $Y$, and suppose that for all $U \in \mathcal{S}$ that $f^{-1}(U)$ is open in $X$. Then $f$ is continuous.

Proof. Use $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ and $f^{-1}(\bigcup U_i) = \bigcup f^{-1}(U_i)$. \qed

Example 1.46. We can describe an interesting topology on $\mathbb{R}$ in this notation. The right half-open interval topology is defined as the topology generated by all intervals $[a, b) \subseteq \mathbb{R}$. The set of all such intervals actually forms a base, not just a subbase, for this topology.

Since $\bigcup_{n=1}^{\infty} (a + 1/n, b) = (a, b)$, the right half-open interval topology is a refinement of the usual topology on $\mathbb{R}$. It is therefore Hausdorff, but we will see later that it is not a metric topology.
1.5 Induced topologies

Definition 1.47. Suppose \((X, \tau)\) is a topological space and that \(A \subseteq X\) is a subset. The \textit{subspace topology} on \(A\) is the topology \(\tau_A\) on \(A\) determined as follows: A set \(U \subseteq A\) is open (resp. closed) in \(A\) if there exists an open (resp. closed) subset \(\tilde{U} \subseteq X\) such that \(\tilde{U} \cap A = U\).

Example 1.48. It’s important to note that if \(A \subset X\) is a non-open subset, and if \(V \subset A\) is open in \(A\), it need not be the case that \(V\) is open in \(X\). For instance, take \([0, 1] \subset \mathbb{R}\). Then \((1/2, 1]\) is open in \([0, 1]\), being \([0, 1] \cap (1/2, \infty)\) for instance, but \((1/2, 1]\) is not open in \(\mathbb{R}\).

Remark 1.49. The subspace topology has an important property. Suppose \(X\) is a topological space and \(A\) a subset of \(X\), given the subspace topology. Suppose \(f : Y \to X\) is a function between topological spaces such that \(\text{im}(f) \subseteq A\). Then the induced function \(f : Y \to A\) is continuous if and only if \(f : Y \to X\) is continuous.

Notation 1.50. Given two topologies \(\tau_1, \tau_2\) on the same set \(X\), we say \(\tau_1\) is \textit{finer} than \(\tau_2\), and \(\tau_2\) is \textit{coarser} than \(\tau_1\), if \(\tau_1 \supseteq \tau_2\). Equivalently, the identity map \(\text{id} : (X, \tau_1) \to (X, \tau_2)\) is continuous.

Construction 1.51. The subspace topology is a special case of a more general construction, that of the induced topology. Here is an instance: suppose \(X\) is a set, and that \(\{Y_i, \tau_i\}_{i \in I}\) is a family of topological spaces, and suppose \(\{f_i : X \to Y_i\}\) is a family of functions. Then the topology on \(X\) \textit{induced} by the \(f_i\) is the coarsest topology, i.e., fewest open sets, such that the \(f_i\) are all continuous.

The subspace topology on \(A \subseteq X\) is the topology induced by the inclusion map \(i : A \to X\).

Definition 1.52. An \textit{embedding} \(i : X \to Y\) is a continuous function such that \(i\) induces a homeomorphism \(X \to i(X)\). An embedding may be an \textit{open embedding} if it has an open image, a \textit{closed embedding} if it has closed image. Some embeddings are neither open nor closed. For instance, the inclusion of a subspace is an embedding, and need not be open or closed.

Construction 1.53. Another special case of the induced topology is the \textit{product topology}. Suppose \(\{(X_i, \tau_i)\}\) is a family of topological spaces. Write \(Y = \prod_{i \in I} X_i\) for the cartesian product of the \(X_i\) as a set, and write \(\pi_i : Y \to X_i\) for the projection. That is, an element \(y \in Y\) is uniquely determined by the values \(\pi_i(y) \in X_i\). The \textit{product topology} on \(Y\) is the topology induced by the maps \(\pi_i\).
Less formally, the product topology on $Y = \prod_{i \in I} X_i$ is the topology generated by the sets $\pi_i^{-1}(U)$ as $i$ ranges over $I$ and $U$ ranges over the open sets of $X_i$. In fact, we can restrict attention to open sets in subbases for the $X_i$, if need be.

**Remark 1.54.** Suppose $(X_1, d_1), \ldots, (X_n, d_n)$ is a finite set of metric spaces. We can put product metrics on $Y = \prod_{i=1}^{n} X_i$, as well as a product topology. The product metrics may be defined in many ways, but for definiteness, we take

$$d((y_1, \ldots, y_n), (x_1, \ldots, x_n)) = \sum_{i=1}^{n} d_i(x_i, y_i).$$

It is an exercise to show that the product metric induces the product topology.

**Remark 1.55.** Given a family of topological spaces $\{(X_i, \tau_i)\}_{i \in I}$, the product set $Y = \prod_{i \in I} X_i$ equipped with the product topology is called the **product space**. It is an example of a universal construction.

Let $Z$ be a topological space. A continuous function $f : Z \to Y$ gives rise, by composition with $\pi_i$, to a family of continuous functions $f_i : Z \to Y \to X_{i}$. The functions $\{f_i\}$ determine the map $f$ uniquely. Moreover, given any family of continuous functions $\{g_i : Z \to X_i\}$, there is a unique continuous function $g : Z \to Y$ such that $g_i = \pi_i \circ g$ for all $i$.

**Example 1.56.** It is important to understand what product topologies look like. Let us start with the common and relatively easy case of two topological spaces: $X$ and $Y$. The product topology on $X \times Y$ is such that the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are both continuous. In fact, it is the coarsest topology with the property that these maps are continuous.

If $U \subseteq X$ is open (resp. closed) then $\pi_1^{-1}(U) \subseteq X \times Y$ is open (resp. closed). A less formal way of writing $\pi_1^{-1}(U)$ is $U \times Y$. Similarly if $V \subseteq Y$ is open, then $X \times V$ is open in $X \times Y$. Putting the two ideas together: $U \times V = U \times Y \cap X \times V$ is an open set in $X \times Y$.

In fact, open sets of this form make up a basis for the product topology. They do not, however, comprise all the open sets in the topology, in general, since $(U_1 \times V_1) \cup (U_2 \times V_2)$ is not generally of the form $U_B \times V_B$.

**Example 1.57.** The real fun with product topologies comes when there is an infinite product of topological spaces. Suppose $X_1, X_2, \ldots$ are countably infinitely many topological spaces (for instance). Write $X = X_1 \times X_2 \times \ldots$. Suppose we have a family $U_i \subseteq X_i$ of open sets in $X_i$. Then $\pi_i^{-1}(U_i) = X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots$. These sets are open in the product topology, as is any finite intersection of them, but there is no reason to expect $U_1 \times U_2 \times U_3 \times \ldots$ to be open unless almost all the $U_i$ equal their respective $X_i$. 
You can get a topology on $\prod_{i \in I} X_i$ by declaring a basis of sets $\prod_{i \in I} U_i$ where $U_i \subseteq X_i$ is open for each $i$. This topology is called the box topology, and it is finer than the product topology.

## 1.6 Coinduced topologies

**Construction 1.58.** Suppose $X$ is a topological space and $f : X \to Y$ is a function. The coinduced topology on $Y$ is the finest topology (most open sets) on $Y$ such that $f$ is continuous. This is especially useful when $f$ is a surjective function, in which case it is called the quotient topology.

The generalization to a family of maps $f_i : X_i \to Y$ is not difficult. One common case of this is when $Y = \bigsqcup_{i \in I} X_i$, the disjoint union of the spaces $X_i$. Then the topology on $Y$ is such that $U \cap Y$ is open if and only if $U \cap X_i$ is open in $X_i$ for all $i$.

**Example 1.59.** One simple, but technically coinduced, topology is as follows. Suppose $(X, \tau)$ and $(Y, \sigma)$ are topological spaces. There are two maps $X \to X \cup Y$ and $Y \to X \cup Y$. The coinduced topology on $X \cup Y$ is the finest topology making both these maps continuous.

As a special case, if $X$ and $Y$ are disjoint, then the coinduced topology has as its open sets all sets $U \cup V$ where $U \subseteq X$ and $V \subseteq Y$ are each open.

**Definition 1.60.** As a special case of the above, if $X$ is a topological space, let $X_+$ denote the disjoint union of $X$ and a point $+$, given the coinduced topology. Observe that a point can have only one topology on it. The open sets of $X_+$ are the sets $U_+$ and $U$ when $U \subseteq X$ is open.

**Example 1.61.** Suppose $(X, \tau)$ is a topological space and $A \subseteq X$ is a subspace. We define the quotient space of $X/A$ in a slightly funny way.

The idea is to collapse all of $A$ to a single point. As a set, $X/A$ is $(X \setminus A)_+$. We define a surjective function from $X_+$ to $X/A$ by sending $+$ to $+$, sending all points in $A$ to $+$ and sending all $x \in X \setminus A$ to $X \setminus A$. Then give $X/A$ the quotient topology for the map $X_+ \to X/A$.

If $A \neq \emptyset$, then we can actually define a surjective map $r : X \to X/A$, but if $A = \emptyset$ then this map is not surjective.

The space $X/A$ has a universal property. Suppose $Y$ is a topological space and $y_0 \in Y$ is a point. Suppose $f : X \to Y$ is a continuous function that also has the property that $f(a) = y_0$ for all $a \in A$. Then there is a unique continuous map $f' : X/A \to Y$ such that $f = f' \circ r$. 
Example 1.62. As a special case of the above, observe that there is a map \( f : [0, 1] \to S^1 \)
given by \( f(x) = (\cos 2\pi x, \sin 2\pi x) \). We will assume the calculus fact that \( f \) is continuous. 
We observe that \( f(0) = f(1) \), so that there is an induced map \( f' : [0, 1]/\{0, 1\} \to S^1 \). This map is also bijective.

With a little thought, we can see that \( f' \) is actually a homeomorphism—this can be done in an even more slick way later.

Example 1.63. Suppose \( X \) is a topological space and \( \sim \subseteq X \times X \) is an equivalence relation 
on \( X \). Then there is a surjective map of sets \( r : X \to X/\sim \), where \( r(x) \) is the equivalence class of \( x \). We endow \( X/\sim \) with the quotient topology, and call it the quotient space of \( X \) by \( \sim \).

The quotient space here has a universal property: if \( f : X \to Y \) is a continuous function so that \( f(x) = f(y) \)
whenever \( x \sim y \), then there exists a unique continuous \( f' : X/\sim \to Y \) such that \( f = f' \circ r \).

This is used especially when there is a group acting on \( X \): say \( G \times X \to X \), and \( x \sim y \)
if there exists \( g \in G \) such that \( gx = y \). Then we abuse notation and write \( X/G \) for the quotient space.

The spaces that arise in this way are frequently not Hausdorff, even when \( X \) is. In particular, we cannot form these quotients without leaving the world of metric spaces.

Exercises

1. Let \( X = \{g, s\} \), and endow \( X \) with the following topology: The subsets \( \emptyset, X, \{g\} \) are open. Give \( [0, 1] \) the usual metric topology.

   (a) Suppose \( f : X \to [0, 1] \) is a continuous function such that \( f(s) = 0 \). Show that \( f(g) = 0 \).

   (b) Produce, with proof, a nonconstant continuous function \( f : [0, 1] \to X \).

2. Let \( X \) be a topological space and let \( A, B \) be two closed subsets of \( X \) such that \( X = A \cup B \). Let \( Y \) be a topological space. Suppose \( f : X \to Y \) is a function such that the restrictions \( f|_A : A \to Y \) and \( f|_B : B \to Y \) are continuous (\( A \) and \( B \) are given the subspace topologies). Prove that \( f \) is continuous.

3. Let \( (X, d) \) be a metric space. Recall that a sequence \( (x_n) \) in \( X \) is said to be a Cauchy sequence if, for all \( \epsilon > 0 \), there exists some \( N_\epsilon \in \mathbb{N} \) such that \( d(x_n, x_m) < \epsilon \) for all \( n, m > N_\epsilon \). The space \( X \) is said to be complete if every Cauchy sequence converges in \( X \). Given an example, with proof, of a homeomorphism \( f : X \to Y \) of metric spaces where \( X \) is complete and \( Y \) is not complete.
1.6. Coinduced topologies
Chapter 2

Closure and sequence methods

2.1 Closure

Definition 2.1. Let $X$ be a topological space and let $A$ be a subset of $X$. The closure of $A$ in $X$, written $\overline{A}$, is the intersection of all closed sets that contain $A$.

Proposition 2.2. The closure operator on subsets of a topological space $X$ has the following properties:

1. $\overline{A}$ is closed.
2. $A \subseteq \overline{A}$, with equality if and only if $A$ is closed.
3. if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$
4. $\overline{\overline{A}} = \overline{A}$.

Proof. 1. Since the intersection of closed subsets is closed, this is immediate.

2. This is the case because $\overline{A}$ is the intersection of closed sets, all containing $A$.

3. Any closed set containing $B$ is a closed set containing $A$. Therefore $\overline{A}$ is the intersection of a family of sets that contains all closed sets containing $B$. The result follows.

4. This is immediate, since $\overline{A}$ is closed.

Proposition 2.3. With notation as before, $x \in \overline{A}$ if and only if, for every open neighbourhood $U \ni x$, the set $U \cap A \neq \emptyset$. 

19
Proof. Consider the statement “every open neighbourhood $U$ of $x$ satisfies $U \cap A \neq \emptyset$”. This is logically equivalent to “for every open set $U$ such that $U \cap A = \emptyset$, the element $x$ is not in $U$”, which is equivalent to “for every closed set $C$ such that $A \subseteq C$, the element $x$ is in $C$” which is equivalent to “$x \in \bar{A}$”. □

Remark 2.4. Given a sequence of inclusions of sets $Z \subseteq Y \subseteq X$, where $X$ is a topological space, it may be the case that the closure of $Z$ in the subspace topology on $Y$ is different from the closure of $Z$ in the topology on $X$. On the other hand, if $Y \subseteq X$ is closed, then the two notions of closure coincide.

2.2 Interior and boundary

Definition 2.5. Suppose $A \subseteq X$. Let $A^\circ$, the interior of $A$, denote the union of all open $U \subseteq A$.

This concept is dual to that of closure. It is immediate that $A^\circ \subseteq A$, with equality if and only if $A$ is open.

Definition 2.6. Suppose $A \subseteq X$. Let $\partial A$, the boundary of $A$, denote $\bar{A} - A^\circ$.

Proposition 2.7. With notation as above, a point $x \in X$ lies in $\partial A$ if and only if every neighbourhood $U \ni x$ satisfies $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$.

Proof. The proof of this is an exercise. □

Proposition 2.8. Let $X$ be a topological space and let $A$ be a subspace. Then there is a division of $X$ into three disjoint subsets: $A^\circ$, $\partial A$ and $(X - A)^\circ$. Moreover $\bar{A} = A^\circ \cup \partial A$.

Proof. The division of $X$ into three parts is really as follows: let $x \in X$ be a point. Then exactly one of the following three cases must obtain:

1. There is some open $U \ni x$ such that $U \subseteq A$. In this case, $x \in A^\circ$.

2. There is some open $U \ni x$ such that $U \subseteq X - A$. In this case, $x \in (X - A)^\circ$.

3. For every open $U \ni x$, both $U \cap A$ and $U \cap X - A$ are not empty. In this case, $x \in \partial X$.

The closure of $A$ consists of those $x$ for which every open neighbourhood meets $A$, by Proposition 2.3. Therefore $\bar{A}$ is the complement of $(X - A)^\circ$. The result follows. □

Corollary 2.9. $\partial A = \partial (X - A)$. 
2.3 Density

Definition 2.10. We say a subset \( A \subseteq X \) is dense if \( \overline{A} = X \). We say \( A \) is sparse or nowhere dense if \( (\overline{A})^o = \emptyset \).

Remark 2.11. The set \( A \) is dense in \( X \) if \( X \) is the only closed set containing \( A \). The contrapositive is that the only open set \( U \subset X \setminus A \) is the empty set. Therefore a set \( A \) is dense if and only if \( U \cap A \) is nonempty whenever \( U \) is a nonempty open subset.

Example 2.12. The subset \( \mathbb{Q} \subset \mathbb{R} \) is dense. On the other hand, the Cantor set \( C \subset [0, 1] \) consisting of all the numbers without a 2 in their (nonterminating) ternary representation is nowhere dense.

Proposition 2.13. Let \( X \) be a topological space, let \( A \subseteq X \) be a dense subset, and let \( Y \) be a Hausdorff topological space. Suppose \( f, g : X \to Y \) are two continuous functions such that \( f(a) = g(a) \) for all \( a \in A \). Then \( f = g \).

In proving this, we use some lemmas that are useful in their own right.

Definition 2.14. Let \( X \) be a topological space. Write \( \Delta \) for the diagonal subset of \( X \times X \) consisting of points \((x, x)\).

Remark 2.15. There is an obvious function \( \Delta : X \to X \times X \), the image of which is the diagonal. This function is continuous (why?) and bijective. It is actually a homeomorphism: consider an open set \( U \subseteq X \), and the projection \( p_1 : X \times X \to X \). Then \( \Delta \cap p_1^{-1}(U) = \Delta(U) \).

Lemma 2.16. Let \( Y \) be a topological space. Then \( Y \) is Hausdorff if and only if the diagonal subset \( \Delta(Y) \subseteq Y \times Y \) consisting of points \((y, y)\) is a closed subset.

Proof. Suppose \( Y \) is Hausdorff. Let \((x, y)\) be a non-diagonal point. Then there exists open \( U \ni x \) and \( V \ni y \) such that \( U \cap V = \emptyset \) and \( U \times V \) therefore gives a subset of \( Y \times Y \) containing \((x, y)\) and disjoint from \( \Delta \). It follows that \( Y \times Y - \Delta \) is open.

Suppose \( \Delta \) is closed. Let \((x, y)\) be a point not in the diagonal. Then there exists some open sets \( U \ni x \) and \( V \ni y \) such that \((U \times Y) \cap (Y \times V)\) does not meet the diagonal. But this implies that \( U \cap V = \emptyset \), as required. \( \square \)

Lemma 2.17. Let \( f, g : X \to Y \) be two continuous functions, and suppose \( Y \) is Hausdorff. The set of points \( x \in X \) such that \( f(x) = g(x) \) is closed in \( X \).

Proof. We can identify \( X \) with \( \Delta \subseteq X \times X \). The function \( (f \times g) : X \times X \to Y \times Y \) is continuous (why?) and the inverse image \( (f \times g)^{-1}(\Delta) \) is therefore closed in \( X \times X \). Therefore \( (f \times g)^{-1}(\Delta) \cap \Delta \) is closed in \( \Delta \), but this consists of precisely those \( x \) such that \( f(x) = g(x) \). \( \square \)
Of course, you can do this much more directly if you like.

\textit{Proof of Proposition.} The set of points where $f(x) = g(x)$ is closed, and contains the dense subset $A$. Therefore it is all of $X$. \hfill \Box

\section*{2.4 Sequences}

I assume you know what a sequence $(x_1, x_2, \ldots)$ in a topological space $X$ means. We’ll denote sequences by $x_n$, or, when it is necessary to specify the indexing variable, $(x_n)_n$.

\textbf{Definition 2.18.} A sequence $(x_1, x_2, \ldots)$ in $X$ converges to $x \in X$ if, for all open $U \ni x$, there exists some $N \in \mathbb{N}$ such that $x_i \in U$ for all $i > N$.

In this formulation, it is clear that the notion of convergence given here is a generalization of the notion you are familiar with from analysis.

Here is a different way of conceptualizing convergence.

\textbf{Notation 2.19.} Unless we say otherwise, the set $\mathbb{N} \cup \{\infty\}$ will be given a topology where $\{n\}$ is open for all $n \in \mathbb{N}$, and a set $U \ni \infty$ is open if and only if it contains some tail: $\{n, n+1, n+2, \ldots\}$.

The induced subspace topology on $\mathbb{N}$ is discrete.

\textbf{Remark 2.20.} We can conceptualize a sequence $x_n$ as a function $x : \mathbb{N} \to X$. Since $\mathbb{N}$ carries the discrete topology, all such functions are continuous.

Suppose this sequence $x : \mathbb{N} \to X$ extends to a continuous function $\hat{x} : \mathbb{N} \cup \{\infty\} \to X$. Then this is equivalent to saying $x_n \to \hat{x}(\infty)$.

\textbf{Remark 2.21.} The subspace $\mathbb{N}$ is dense in $\mathbb{N} \cup \{\infty\}$.

In this formulation, the following is an immediate consequence of the fact that composites of continuous functions are continuous.

\textbf{Proposition 2.22.} Let $x_n$ be a sequence in $X$ and suppose $x_n$ converges to $x$. Let $f : X \to Y$ be a continuous function. Then $f(x_n)$ converges to $f(x)$ in $Y$.

\textbf{Remark 2.23.} Similar tricks with composite functions $\mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\}$ can be used to show that if $x_n \to x$ and if $y_n$ is a subsequence of $x_n$, then $y_n \to x$. You can also do this directly. You probably get enough of this sort of thing in analysis lectures.

We know from analysis lectures that the limit of a convergent sequence is unique in a metric space. Unfortunately, this does not generalize to non-Hausdorff spaces.
Example 2.24. Consider an infinite set $X$ with the cofinite topology. Let $x_n$ be a sequence in $X$ in which $x_i \neq x_j$ for all $i \neq j$. Such a sequence exists because there exists an injective map $\mathbb{N} \to X$.

Let $y \in X$. Consider any open $U \ni y$. The set $X \setminus U$ consists of only finitely many elements, and therefore only finitely many elements of $(x_n)$ lie outside $U$. In particular, some tail of the sequence $(x_n)$ lies entirely inside $U$, and so $x_n \to y$. But $y$ was arbitrary.

This shows that even in a $T_1$ topological space, limits of sequences may not be unique.

**Proposition 2.25.** Let $X$ be a Hausdorff topological space and let $x_n$ be a sequence in $X$. Suppose $x_n \to y$ and $x_n \to z$. Then $y = z$.

**Proof.** The two limits can be encoded as continuous functions

$$
\begin{array}{ccc}
N \cup \{\infty\} & \xrightarrow{\hat{y}} & X \\
\downarrow & \swarrow & \downarrow \\
N & \xleftarrow{\hat{z}} & X
\end{array}
$$

Here $\hat{y}(0) = y$ and $\hat{z}(0) = z$, whereas $\hat{y}|_N = \hat{z}|_N$. Since $N$ is dense in $N \cup \{\infty\}$, and $X$ is Hausdorff, it follows that $\hat{y} = \hat{z}$. \qed

Of course, one can prove this in a lower-level way, by working directly with points and sets. You have surely all done this in analysis courses.

Our function-based approach to sequence convergence gives us a slick proof of the following:

**Proposition 2.26.** Let $\{X_i\}_{i \in I}$ be a family of topological spaces, let $X$ denote the product. Let $(x_j)$ be a sequence in $X$. Let $y \in X$ be an element. Then $(x_j) \to y$ if and only if $\pi_i(x_j) \to \pi_i(y)$ for all $i$.

That is, a sequence in a product space converges to $y$ if and only if all the projections converge to the appropriate projections of $y$.

In a metric space, the closed sets admit a characterization in terms of limits of sequences. This carries over to all first-countable spaces.

**Definition 2.27.** Let $X$ be a topological space and $A$ a subspace. Say that $A$ is sequentially closed if it has the following property: if $(a_n)_n$ is a sequence in $A$ that converges to $x \in X$, then $x \in A$. 
Proposition 2.28. Let $X$ be a topological space and $A$ a closed subspace. Then $A$ is sequentially closed.

Proof. Consider the commutative diagram of continuous maps

$$
\begin{array}{ccc}
\mathbb{N} & \longrightarrow & \mathbb{N} \cup \{\infty\} \\
\downarrow & & \downarrow \hat{x} \\
A & \longrightarrow & X
\end{array}
$$

which implements $(a_n)_n \to x$. If $A$ is closed in $X$, then $\hat{x}^{-1}(A)$ is closed in $\mathbb{N}$. Since $\hat{x}^{-1}(A) \supseteq \mathbb{N}$, it follows that $\hat{x}^{-1}(A) \supseteq \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. □

Proposition 2.29. Suppose $X$ is a first-countable topological space and $A$ is a sequentially closed subspace. Then $A$ is closed.

The proof will be an exercise.

Corollary 2.30. Let $X$ and $Y$ be topological spaces where $X$ is first countable. Suppose $f : X \to Y$ is a function with the property that $x_n \to x$ in $X$ implies $f(x_n) \to f(x)$ in $Y$. Then $f$ is continuous.

Proof. Let $A \subseteq Y$ be a closed set. We will show $f^{-1}(A)$ is closed. If $A$ is empty, there is nothing to show, so assume it is not. To show $f^{-1}(A)$ is closed, it is sufficient to show it is sequentially closed. Let $x_n$ in $f^{-1}(A)$ be a sequence converging to $x \in X$. Then $f(x_n)$ is a sequence in $A$ converging to $f(x)$, and since $A$ is closed, $f(x) \in A$, which implies $x \in f^{-1}(A)$. □

2.5 Completions of Metric Spaces

Definition 2.31. A sequence $x_n$ in a metric space $(X, d)$ is said to be Cauchy if, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > N$.

Remark 2.32. All convergent sequences are Cauchy. The sequence $x_n = \sum_{i=1}^{n} 1/i$ is not Cauchy in $\mathbb{R}$—even though the distance between successive terms tends to 0.

Definition 2.33. A metric space $(X, d)$ is complete if every Cauchy sequence converges in $X$.

Example 2.34. It is well known that $\mathbb{R}^n$ is complete. A product of two metric spaces is complete. A subset of a complete metric space is complete if and only if it is closed.
Chapter 2. Closure and sequence methods

Example 2.35. The metric space \( \mathbb{R} \) is homeomorphic to any bounded open interval. For instance \( f(x) = \frac{x}{\sqrt{1 + x^2}} \) is a homeomorphism \( f : \mathbb{R} \to (-1, 1) \). This shows that completeness is not a topological, but rather a metric, property.

Definition 2.36. A map of metric spaces \( f : X \to Y \) is an isometry if \( d(f(x), f(x')) = d(x, x') \) for all \( x, x' \in X \). We remark that an isometry is necessarily injective.

Definition 2.37. A pseudometric space \((X, \delta)\) is a set \( X \) equipped with a function \( \delta : X \times X \to [0, \infty) \) satisfying the axioms of a metric space except that \( \delta(x, y) = 0 \) does not necessarily imply \( x = y \).

Proposition 2.38. Let \((X, d)\) be a metric space. Let \( CX \) denote the set of all Cauchy sequences in \( X \), and let \((x_n), (y_n) \in CX\). Then the sequence \( d(x_n, y_n) \) converges in \([0, \infty)\). If we define \( \delta((x_n), (y_n)) = \lim_{n \to \infty} d(x_n, y_n) \), then \( \delta \) is a pseudometric on \( CX \).

Sketch of proof. We claim \( d(x_n, y_n) \) is a Cauchy sequence in \( \mathbb{R} \). For any \( \epsilon > 0 \), choose \( N \) sufficiently large so that \( d(x_n, x_m) < \epsilon/2 \) and \( d(y_n, y_m) < \epsilon/2 \) for all \( n, m > N \). Then \( d(x_n, y_n) \leq d(x_n, y_m) + \epsilon \) by the triangle inequality. By a symmetric argument \( |d(x_n, y_n) - d(x_m, y_m)| < \epsilon \).

Since \( \mathbb{R} \) is complete, \( \delta \) is well defined. The pseudometric properties are straightforward to prove:

\[
\lim_{n \to \infty} d(x_n, y_n) \leq \lim_{n \to \infty} d(x_n, z_n) + \lim_{n \to \infty} d(z_n, y_n)
\]

establishes the triangle inequality, for instance. \( \square \)

Remark 2.39. If \((x_n)\) is a Cauchy sequence and \((x_{n_i})\) is a subsequence, then \( \delta((x_n), (x_{n_i})) = 0 \). This follows from the Cauchy property.

Construction 2.40. The relation \( x \sim y \) if \( \delta(x, y) = 0 \) is an equivalence relation (use the triangle inequality). Write \( QX \) for the set of equivalence classes. If \( \delta(x, x') = 0 \) then the triangle inequality shows that \( \delta(x, y) = \delta(x', y) \). Therefore \( \delta \) induces a well-defined metric, also denoted \( \delta \) here, on \( QX \).

Proposition 2.41. Let \((X, d)\) be a metric space. The metric space \((QX, \delta)\) is complete.

Sketch of proof. Let \((x^n)\) denote a Cauchy sequence in \( QX \). We can choose representatives of each \( x^n \) so that \( d(x^n_i, x^n_j) < \max\{1/i, 1/j\} \) for all \( i, j \) — pass to a subsequence if need be. Define a sequence by \( z_n = x^n_n \). We claim that \( z_n \) a Cauchy sequence and that \( x^n \to z \). First we remark that for any given \( n \):

\[
d(x^n_i, x^n_j) \leq d(x^n_i, x^n_i) + d(x^n_i, x^n_j) + d(x^n_j, x^n_n) \leq \delta(x^n_i, x^n_j) + 2/n
\]
so that
\[d(z_n, x^n_i) \leq d(x^n_n, x^n_i) + d(x^n_i, x^n_j) \leq \delta(x^n_n, x^n_i) + \max\{1/i, 1/n\} + 2/n.\]

But since \((x^n)_n\) is Cauchy, it follows easily that \(z = (z_n)_n\) is a Cauchy sequence. Moreover
\[\delta(z, x^n_i) \leq \delta(x^n_i, x^n_j) + \max\{1/i, 1/j\} + 2/j\]
for all \(j\). This shows that \(x^n_i \to z\). □

**Proposition 2.42.** Define a function \(\iota : X \to QX\) by \(\iota(x) = (x, x, \ldots)\). Then \(\iota\) is an isometry with dense image.

**Proof.** That \(\iota\) is an isometry is trivial. The density of the image is proved by observing that if \((x_n) = (x_1, x_2, \ldots)\) represents an element of \(QX\) then \((\iota(x_1), \iota(x_2), \ldots)\) converges to \((x_n)\). □

**Remark 2.43.** The space \(QX\) as constructed in this proof is called the completion of \(X\). We have shown that any metric space embeds isometrically as a dense subset of a complete metric space.
Chapter 3

Compactness

3.1 Elementary Theory

**Definition 3.1.** An *open cover* \( U \) of a topological space \( X \) is a collection \( \{U_i\}_{i \in I} \) of open sets such that \( \bigcup_{i \in I} U_i = X \).

**Definition 3.2.** A topological space \( X \) is *compact* if every open cover \( \{U_i\}_{i \in I} \) contains a finite subcover \( \{U_1, \ldots, U_n\} \) such that \( X = \bigcup_{i=1}^n U_i \).

**Remark 3.3.** Sometimes the term “quasicompact” is used if \( X \) is not Hausdorff. This is the normal usage in algebraic geometry, which is unfortunate since this is also where one sees most non-Hausdorff compact spaces. In French “compact” means what “compact Hausdorff” means in English.

**Proposition 3.4.** Let \( f : X \to Y \) be a continuous function and suppose \( X \) is compact. Then \( f(X) \) is a compact subspace of \( Y \).

**Proof.** Let \( \{U_i\}_{i \in I} \) be an open cover of \( f(X) \). Consider \( \{f^{-1}(U_i)\}_{i \in I} \), which is an open cover of \( X \), and therefore has a finite subcover \( \{f^{-1}(U_1), \ldots, f^{-1}(U_n)\} \). The set \( \{U_1, \ldots, U_n\} \) is the required finite subcover of \( f(X) \). \( \square \)

**Proposition 3.5.** If \( X \) is a space and \( C_1 \) and \( C_2 \) are two compact subsets, then \( C_1 \cup C_2 \) is compact.

**Proof.** A cover of \( C_1 \cup C_2 \) contains a finite subset covering \( C_1 \) and a finite subset covering \( C_2 \). \( \square \)

**Proposition 3.6.** Suppose \( C \) is a closed subspace of a compact space \( X \). Then \( C \) is compact.
3.2. The Tube Lemma

Proof. Consider an open cover \( \{U_i\}_{i \in I} \) of \( C \). Let \( V_i \) be open in \( X \) and satisfy \( U_i = C \cap V_i \). Consider \( \{V_i\}_{i \in I} \cup \{X \setminus C\} \). This is an open cover of \( X \). Any finite subcover induces a finite subcover of \( \{U_i\} \). \( \square \)

**Proposition 3.7.** Suppose \( C \) is a compact subspace of a Hausdorff space \( X \). Then \( C \) is closed in \( X \).

Proof. Let \( x \in X \setminus C \). For each \( y \in C \) we can find disjoint open neighbourhoods \( U_y \ni y \) and \( V_y \ni x \). Finitely many of the \( U_y \) suffice to cover \( C \), since it is compact, and therefore there exists an intersection of finitely many \( V_y \) that is disjoint from \( C \). But a finite intersection of open sets is open. This proves that \( x \) has an open neighbourhood disjoint from \( C \). \( \square \)

The following corollary is extremely useful.

**Corollary 3.8.** Let \( f : X \to Y \) be a continuous bijection between topological spaces where \( X \) is compact and \( Y \) is Hausdorff. Then \( f \) is a homeomorphism.

Proof. We prove that \( f \) is a closed map. This implies that the closed subsets of \( X \) are in bijective correspondence with the closed subsets of \( Y \). Let \( C \subseteq X \) be closed, then \( C \) is compact, so \( f(C) \) is compact, so \( f(C) \) is closed. \( \square \)

**Proposition 3.9.** A compact Hausdorff space \( X \) is a \( T_4 \) space.

Proof. In a time-honoured tradition, we prove that \( X \) is \( T_3 \). The proof that it’s \( T_4 \) is the same argument again.

Let \( p \) be a point and \( C \) be a closed subset disjoint from \( p \). Since \( X \) is compact, \( C \) is compact. Since \( X \) is Hausdorff, for each \( c \in C \) we can find disjoint \( U_c \ni p \) and \( V_c \ni c \) such that \( U_c \cap V_c = \emptyset \). Finitely many \( V_c \) suffice to cover \( C \): say \( V(p) = V_{c_1} \cup \cdots \cup V_{c_n} \ni C \). Then \( U(p) = \bigcap_{i=1}^n U_{c_i} \) is an open set disjoint from \( V(p) \) and \( p \in U(p) \).

If \( C_1 \) and \( C_2 \) are disjoint closed sets, for each \( p \in C_1 \) we can find disjoint open \( U(p) \ni p \) and \( V(p) \ni C_2 \). Finitely many of the \( U(p) \) suffice to cover \( C_1 \) and we use the same finite-intersection idea again to prove \( X \) is normal. \( \square \)

3.2 The Tube Lemma

**Lemma 3.10** (Generalized Tube Lemma). Let \( X \) and \( Y \) be topological spaces and \( A \subseteq X \) and \( B \subseteq Y \) be compact subsets. If \( N \) is an open subset of \( X \times Y \) containing \( A \times B \), then there exists open subsets \( U \subseteq X \) and \( V \subseteq Y \) such that \( A \times B \subseteq U \times V \subseteq N \).
Proof. If $A$ is empty, there is nothing to show.

Let $a \in A$. We produce a cover of $a \times B$ as follows. For each $b \in B$, we can find an open set of the form $U_b \times V_b$ such that $(a, b) \in U_b \times V_b \subseteq N$. By compactness, there is a finite set $\{(a, b_1), (a, b_2), \ldots, (a, b_n)\}$ of such points so that the associated $U_{b_i} \times V_{b_i}$ form a cover $A \times B$. If we take $U(a) = \bigcap_{i=1}^n U_{b_i}$ and $V(a) = \bigcup_{i=1}^n V_{b_i}$, then the open set $U(a) \times V(a)$ has the following properties:

1. It is contained in $N$.
2. It contains $\{a\} \times B$.

We now repeat this procedure for all $a \in A$ to produce a cover $\{U(a) \times V(a)\}_{a \in A}$ of $A \times B$ that is contained in $N$. Since $A$ is compact, we can find a finite set $\{a_1, a_2, \ldots, a_r\}$ of points in $A$ so that $\bigcup_{i=1}^r U(a_i) \supseteq A$. Define $U = \bigcup_{i=1}^r U(a_i)$ and $V = \bigcap_{i=1}^r V(a_i)$. Then $A \subseteq U$, by construction, and $B \subseteq V$, since $B \subseteq V(a)$ for all $a$. The set $U \times V$ is open and contains $A \times B$. It remains to verify that it is contained in $N$.

Suppose $(x, y) \in U \times V$. Then $x \in U(a_i)$ for at least one of the $a_i$ chosen above, and $y \in V \subseteq V(a_i)$. Therefore $(x, y) \in U(a_i) \times V(a_i) \subseteq N$. This proves the required containment. \qed

Corollary 3.11 (The Tube Lemma). Let $X$ and $Y$ be topological spaces and suppose $X$ is compact. Let $y \in Y$. Suppose $N$ is an open neighbourhood of $X \times \{y\}$, then there exists an open $U \ni y$ such that $X \times U \subseteq N$.

This implies the following weak version of Tychanoff’s theorem, Theorem 3.26.

Corollary 3.12. Let $X$ and $Y$ be compact topological spaces. Then $X \times Y$ is compact.

Proof. Suppose $\mathcal{U} = \{U_i\}$ is an open cover of $X \times Y$. For each $y \in Y$, the set $X \times \{y\}$ is compact, and therefore there exists a finite subset $\mathcal{U}_y$ of $\mathcal{U}$ such that $\bigcup \mathcal{U}_y \supseteq X \times \{y\}$. By use of the tube lemma, with $N = \bigcup \mathcal{U}_y$, we can find an open $V_y \ni y$ such that $X \times V_y \subseteq \bigcup \mathcal{U}_y$.

Since $Y$ is compact, finitely many $V_y$ suffice to cover $Y$: say, $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$ for some points $y_1, \ldots, y_n \in Y$. Now take the finite union of finite sets of open sets in $X \times Y$:

$$\mathcal{W} = \bigcup_{i=1}^n \mathcal{U}_{y_i}.$$ 

This is a finite subcover of $\mathcal{U}$. \qed
3.3 Compactness in Metric Spaces

The Lebesgue Covering Lemma

**Definition 3.13.** If $X$ is a metric space and $A$ is a subspace of $X$, then the *diameter* of $A$ is the supremum of the set $\{d(x, y) \mid x, y \in A\}$. It is finite if and only if $A$ is bounded.

**Lemma 3.14 (Lebesgue covering).** Let $X$ be a compact metric space and let $\mathcal{U}$ be an open cover of $X$. There exists $\delta > 0$ such that whenever $A$ is a subset of $X$ of diameter less than $\delta$, there exists some $U_i \in \mathcal{U}$ such that $A \subseteq U_i$.

**Proof.** For each $x \in X$, choose some $U_x \in \mathcal{U}$ containing $x$, and then choose some radius $r_x > 0$ such that $B(x, r_x) \subseteq U_x$.

Now consider the balls $B(x, r_x/2)$. These are open, and they form an open cover of $X$, so we may select a finite set of such balls that cover $X$, say:

$$X = B(x_1, r_{x1}/2) \cup B(x_2, r_{x2}/r) \cup \cdots \cup B(x_n, r_{xn}/2).$$

Let $r$ denote the least of the radii $r_1, r_2, \ldots, r_n$, and set $\delta = r/2$.

Let $A$ be a nonempty set of diameter $\delta$ or less. Let $x \in A$ be a point. Then $x \in B(x_i, r_i/2)$ for some $i$. Suppose $y \in A$ is another point. Then $d(y, x_i) < d(y, x) + d(x, x_i) = \delta + r_i/2 \leq r_i$. We have shown that $A \subseteq B(x_i, r_i) \subseteq U_x$, as required. \(\Box\)

The Heine–Borel Theorem

The Heine–Borel Theorem says that a subspace $C \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded. This section is devoted to a generalization of this fact, Theorem 3.24.

Before we turn to the theorem itself, we note some consequences

**Lemma 3.15.** Let $X$ be a compact topological space and $f : X \to \mathbb{R}$ be a continuous function. Then $f$ attains a maximum value on $X$.

**Proof.** The image, $f(X)$, is a compact subset of $\mathbb{R}$ and is therefore closed and bounded. \(\Box\)

**Example 3.16.** The quotient $[0, 1]/\{0, 1\}$ is homeomorphic to $S^1$. We can now prove this quickly. The interval $[0, 1]$ is compact, so its surjective image $[0, 1]/\{0, 1\}$ is also compact. Therefore the map $f : [0, 1]/\{0, 1\} \to S^1$ given by $(\cos 2\pi \theta, \sin 2\pi \theta)$ is a continuous bijection with compact source and Hausdorff target.
**Definition 3.17.** A topological space $X$ is said to be **sequentially compact** if every sequence has a convergent subsequence.

Sequential compactness is neither stronger than nor weaker than compactness. For metric spaces, however, we will prove that the two concepts coincide.

**Definition 3.18.** A metric space $X$ is said to be **totally bounded** if, for all $\epsilon > 0$, one can cover $X$ by finitely many balls $B(x, \epsilon)$.

**Remark 3.19.** A totally bounded subspace of a metric space is necessarily bounded. Conversely, a bounded subspace of $\mathbb{R}^n$ is totally bounded. In infinite-dimensional spaces, however, bounded may not imply totally bounded. For instance, the unit ball $B(0, 1)$ of $\ell_2$ is not totally bounded.

**Proposition 3.20.** A compact metric space is totally bounded.

**Proof.** For any $\epsilon$, the balls $B(x, \epsilon)$ form an open cover. The finite-subcover property implies total boundedness. □

**Proposition 3.21.** A compact metric space is complete.

**Proof.** Suppose $X$ is a compact metric space. There is an isometric embedding $\iota: X \to QX$ where $QX$ is complete and $\iota(X)$ is dense. Since $\iota(X)$ is compact, it is also closed. Therefore $\iota$ is a bijective isometry (a metric equivalence) and $X$ is complete. □

**Proposition 3.22.** A complete, totally bounded metric space $X$ is sequentially compact. In particular, a compact metric space is sequentially compact.

**Proof.** Let $X$ be a totally bounded metric space. Let $(x_n^0)_n$ be a sequence. We produce a Cauchy sequence recursively. Cover $X$ by finitely many balls of radius $1$. One of these, $B_1$, contains a tail of the sequence. Let $x^1$ be the sequence that starts with $x_0^0$ and thereafter consists of only those terms in $B_1$.

Cover $B_1$ by finitely many balls of radius $1/2$. One of these, $B_2$, contains a tail of the sequence $x^1$. Form the sequence $x^2 = (x_1^1, x_2^1, \text{subsequent terms in } B_2)$

Cover $B_2$ by finitely many balls of radius $1/3$. One of these, $B_3$, contains a tail of the sequence $x^2$. Form the sequence $x^3 = (x_1^2, x_2^2, x_3^2, \text{subsequent terms in } B_3)$
3.3. Compactness in Metric Spaces

Note that the sequences $x^i$ stabilize: $x^i_n = x^{i+1}_n$ if $n \leq i$. Form $x^\infty$ as $(x^i_n)_n$. Observe that by virtue of how we constructed this, if $n < m$, then $x^\infty_n$ and $x^\infty_m$ lie in the same $1/n$-ball, so that $x^\infty$ is a Cauchy subsequence of $(x^i_n)$.

Since $X$ is complete, this Cauchy subsequence converges and so $x^0$ has a convergent subsequence. □

Proposition 3.23. A sequentially compact metric space is complete and totally bounded.

Proof. Suppose $X$ is sequentially compact. If $(x_n)$ is a Cauchy sequence, then $x_n$ converges to the limit of any convergent subsequence. Therefore $X$ is complete.

If $X$ is not totally bounded, we can find some $\epsilon > 0$ such that $X$ cannot be covered by $\epsilon$-balls. Therefore there is an infinite sequence $x_n$ of points that are pairwise at distance at least $\epsilon$ from each other. This sequence can have no Cauchy subsequence and therefore no convergent subsequence, a contradiction. □

Theorem 3.24. Let $(X, d)$ be a metric space. The following are equivalent:

1. $X$ is compact,
2. $X$ is complete and totally bounded,
3. $X$ is sequentially compact.

Proof. We have already proved that 1 implies 2 and that 2 is equivalent to 3. Let us now assume that $X$ is sequentially compact (and therefore totally bounded). Let $\{U_i\}_{i \in I}$ be a cover. We claim that for some $n \in \mathbb{N}$, each ball $B(x, 1/n)$ is contained in some $U_i$.

Suppose for the sake of contradiction that this is not the case. Then let $x_n \in X$ be a sequence such that $B(x_n, 1/n)$ is not contained in any $U_i$. The sequence $(x_n)_n$ contains a subsequence converging, to some limit $x$. This $x$ is in some $U_i$, and furthermore, there is some $r > 0$ such that $B(x, 2r) \subseteq U_i$ and such that $B(x, r)$ contains infinitely many terms of $(x_n)$. But then for any $n > 1/r$, we have $B(x_n, 1/n) \subseteq U_i$, a contradiction.

Therefore the claim holds. Now, fix a radius $1/n$ such that each of the balls $B(x, 1/n)$ is contained in some $U_i$, depending on $x$. Since $X$ is totally bounded, finitely many such balls suffice to cover $X$, and therefore finitely many of the $U_i$ suffice to cover $X$. So $X$ is compact. □

Corollary 3.25. A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.
3.4 Tychanoff’s Theorem

Theorem 3.26 (Tychanoff’s Theorem). Suppose \( \{X_i\}_{i \in I} \) is a family of compact topological spaces. Then the product space \( \prod_{i \in I} X_i \) is compact.

Remark 3.27. This theorem is equivalent to the axiom of choice (in the generality in which it has been stated). In the case of a product of two compact spaces, \( X \times Y \), the theorem is much simpler to prove. This is the most useful form of the theorem, too. But we’ll just do the general version.

Lemma 3.28. Let \( X \) be a topological space and suppose that \( X \) is not compact. Then there exists a cover \( \{U_i\}_{i \in I} \) of \( X \) that does not have a finite subcover and that is maximal with this property. I.e., for any open set \( V \notin \{U_i\}_{i \in I} \), the open cover \( \{U_i\}_{i \in I} \cup \{V\} \) has a finite subcover.

Proof. We apply Zorn’s lemma. Suppose \( \{U_j\}_{j \in J} \) is a chain of open covers, each without a finite subcover. Then \( \mathcal{V} = \bigcup_{j \in J} U_j \) is an open cover of \( X \). Suppose \( \mathcal{V} \) has a finite subcover \( \{U_1, U_2, \ldots, U_n\} \). Then there is some \( U_j \) containing all these sets, a contradiction.

Since any chain of covers without finite subcovers has an upper bound, there must be a maximal such chain by Zorn’s lemma. \( \square \)

Theorem 3.29 (Alexander’s subbase theorem). Let \( X \) be a topological space and let \( S \) be a subbase for the topology on \( X \), such that \( \bigcup S = X \). The space \( X \) is compact if and only if every cover \( \{S_i\}_{i \in I} \subseteq S \) has a finite subcover.

Proof. One direction is trivial: \( X \) is compact then a fortiori every subbasic cover has a finite subcover.

Suppose therefore for the sake of contradiction that every cover drawn from \( S \) has a finite subcover, but that \( X \) is nonetheless not compact. Let \( \mathcal{U} = \{U_i\}_{i \in I} \) be a maximal open cover without a finite subcover. Consider \( \mathcal{U} \cap S \). This cannot form a cover of \( X \), so there is some \( x \in X \) such that \( x \) is not contained in any of the sets of \( \mathcal{U} \cap S \). Nonetheless, there exists some \( V \in \mathcal{U} \) containing \( x \). We can write \( x \in S_1 \cap S_2 \cap \cdots \cap S_n \subseteq V \) where \( S_i \in S \) for all \( S \). Because of the way we chose \( x \), none of the \( S_i \) can appear in \( \mathcal{U} \), so that each of the covers \( \mathcal{U} \cup \{S_i\} \) strictly contains \( \mathcal{U} \), and therefore each contains a finite subcover of \( X \). These subcovers must involve \( S_i \). There are therefore finite subcovers

\[
\{U_{1,1}, U_{1,2}, \ldots, U_{1,m_1}, S_1\} \\
\{U_{2,1}, U_{2,2}, \ldots, U_{2,m_2}, S_2\} \\
\vdots \\
\{U_{n,1}, U_{n,2}, \ldots, U_{n,m_n}, S_n\}
\]
where the open sets \( U_{i,j} \in \mathcal{U} \). But then \( \bigcup_{i,j} U_{i,j} \cup V \) is a union of open sets in \( U \) and it contains every point in \( X \), a contradiction. \( \square \)

**Proof of Tychanoff’s theorem.** The product topology \( X = \prod_{i \in I} X_i \) has a subbase given by all sets of the form \( \pi_i^{-1}(U) \) where \( U \) is open in \( X_i \). By virtue of Alexander’s subbase theorem, it is sufficient to prove that any cover of \( X \) by sets from this subbase has a finite subcover.

Suppose \( \mathcal{S} \) is an open cover of \( X \) where the open subsets are taken from the subbase above. For any coordinate \( i \), consider the set \( \mathcal{S} \cap \{ \pi_i^{-1}(U) \mid U \subseteq X_i \} \). We claim that for at least one \( i \), the sets \( U \) appearing here must form an open cover of \( X_i \). Suppose not, then in each \( X_i \), there exists some \( x_i \) which is not in any of the appropriate open \( U \subset X_i \). But now consider \( x \in X \) such that \( \pi_i(x) = x_i \) for all \( i \). This does not lie in any set in \( \mathcal{S} \), a contradiction. Hence the claim is proved.

We may assume that there is some \( X_i \) such that the open sets \( \pi_i^{-1}(U_j) \) appearing in \( \mathcal{S} \) are such that the \( U_j \) cover \( X_i \). Since \( X_i \) is compact, we can take \( U_1, \ldots, U_n \) that cover \( X_i \). Then \( \pi_i^{-1}(U_1), \ldots, \pi_i^{-1}(U_n) \) cover \( X \). \( \square \)

### 3.5 Compactifications

**Definition 3.30.** Let \( X \) be a topological space. A **compactification** of \( X \) is an embedding \( i : X \subset \hat{X} \) where \( \hat{X} \) is a compact space and where the image of \( i \) is dense in \( \hat{X} \).

**Remark 3.31.** If we require both \( X \) and \( \hat{X} \) to be Hausdorff, which is reasonable if we are doing geometry, and if \( X \) is already compact, then the only available compactification is \( X \) itself, up to homeomorphism. If \( X \) is not compact, however, then there may be many different compactifications.

**Remark 3.32.** If \( i : X \to \hat{X} \) is a compactification, then we may refer to \( \hat{X} \) as “the compactification”, especially when the map \( i \) is obvious. We may say the compactification has a property \( P \), if the space \( \hat{X} \) has that property. For instance, a “Hausdorff compactification” is a compactification of \( i : X \to \hat{X} \) in which \( \hat{X} \) is Hausdorff.

**Construction 3.33.** Let \( X \) be a topological space. Let the **one-point compactification** \( X \to X \cup \{\infty\} \) be the inclusion of \( X \) into a space \( X \cup \{\infty\} \) consisting of \( X \) itself and a new point “at infinity”. The topology on \( X \cup \{\infty\} \) is defined as follows:

- open subsets of \( X \) are open in \( X \cup \{\infty\} \).
- A subset \( U \ni \{\infty\} \) is open if \( X \setminus (U \cap X) \) is a closed compact subset of \( X \).
We note that this is actually a topology. Verifying this is routine.

**Remark 3.34.** In spite of the name, this is not a compactification if \( X \) is already compact. In that case, \( X \cup \{\infty\} = X_+ \).

**Remark 3.35.** If \( X \) is Hausdorff, then all compact subsets of \( X \) are closed, so the “closed” in “closed compact” is redundant. This is the most commonly-used case of the construction.

**Proposition 3.36.** If \( X \) is not compact, then the obvious inclusion \( i : X \rightarrow X \cup \{\infty\} \) is a compactification.

**Proof.** First we observe that \( i : X \rightarrow X \cup \{\infty\} \) is an embedding. It is clearly injective and easy to verify that it is continuous. In fact, it is an open map so it follows that it is an embedding.

Next we prove that \( X \cup \{\infty\} \) is compact. Suppose \( \{U_i\}_{i \in I} \) is a cover. Then at least one \( U_i \) contains \( \infty \), so that \( X \setminus \{U_i\} \) is compact. Therefore finitely many of the other \( U_i \) suffice to cover \( X \setminus U_1 \).

Finally, we verify that \( X \subseteq X \cup \{\infty\} \) is dense. The space \( X \) is not compact itself, so that \( \{\infty\} \) is not an open set. Since \( X \) is not closed, so its closure must be \( X \cup \{\infty\} \).

**Example 3.37.** We have already seen a one-point compactification: \( \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\} \) was used to define convergence of a sequence.

**Example 3.38.** Here is another, more troubling, example. Take \( \mathbb{Q} \) with the usual topology and form the one-point compactification \( \mathbb{Q} \rightarrow \mathbb{Q} \cup \{\infty\} \). The open neighbourhoods of \( \infty \) are the complements of compact sets. We claim that no compact set \( K \) contains the intersection of \( \mathbb{Q} \) with an open interval \( I \). If it did, we could find a sequence in \( I \cap \mathbb{Q} \) converging to an irrational number, and this sequence could have no convergent subsequence in \( \mathbb{Q} \).

Therefore, for any \( q \in \mathbb{Q} \), every open neighbourhood \((a, b) \cap \mathbb{Q} \) meets every open neighbourhood of \( \infty \). We deduce that \( \mathbb{Q} \cup \{\infty\} \) is compact, but it is not Hausdorff.

Nonetheless, something funny happens in this space about the limits of convergent sequences—they are still unique even though the space is not Hausdorff. The space has other properties that make it like a Hausdorff space. The space is weak Hausdorff: this will be defined in the homework.

Now we devote ourselves to finding conditions that ensure that the one-point compactification of a Hausdorff space is again Hausdorff.

**Definition 3.39.** We will say a space \( X \) is locally compact if every point \( x \in X \) is contained in an open set \( U \) that is itself contained in a compact set \( K \).
Proposition 3.40. If $X$ is a Hausdorff space then the following are equivalent:

1. $X$ is locally compact,
2. every point $x \in X$ has an open neighbourhood $U$ such that $\bar{U}$ is compact,
3. every point $x \in X$ has a local base consisting of open sets $U$ such that $\bar{U}$ is compact.

Proof. It is trivial that (iii) implies (ii) which implies (i). Therefore it suffices to show that (i) implies (iii). Let $x \in X$ be a point and choose $U \subset K$ such that $x \in U$ and $U \subseteq K$ where $U$ is open and $K$ is compact. Let $\{V_i\}_{i \in I}$ be any local base at $x$. Then $\{U \cap V_i\}_{i \in I}$ is another local base at $x$ and, furthermore, each of these sets is contained in $K$, which is closed—due to the Hausdorff property. Therefore their closures are closed and contained in $K$. Consequently, their closures are compact. □

Proposition 3.41. Let $X$ be a locally compact Hausdorff space. Then the one-point compactification $X \cup \{\infty\}$ is Hausdorff.

Proof. Let $x \neq y$ be two points in $X \cup \{\infty\}$. These two points have disjoint open neighbourhoods in $X$ if they both lie in $X$. Therefore the only case we have to check is when $y = \{\infty\}$. We can find an open $U \ni x$ such that the closure in $X$, denoted $\bar{U}$ is compact, and so $X \cup \{\infty\} \setminus \bar{U}$ is an open neighbourhood of $\{\infty\}$ disjoint from $U$. □

Lemma 3.42. Let $i : X \to Y$ be any Hausdorff compactification of a locally compact Hausdorff space. Then $i$ is an open map.

Proof. We prove that $i(X)$ is open in $Y$. Let $x \in X$. There exists some open $U \ni x$ and a compact $U \subseteq K$ in $X$. We know that $i(K)$ is compact, and hence closed, in $Y$. Since $i(X)$ is homeomorphic to $X$, there exists an open $W \ni i(x)$ such that $W \cap i(X) = i(U)$. Now consider $W \setminus i(K)$. This is an open set in $Y$ and $W \cap i(X) \subseteq i(K)$ implies that it is disjoint from $i(X)$. Since $i(X)$ is dense in $Y$, this means that it must be empty, so $W \subseteq i(K) \subseteq i(X)$. It follows $i(X)$ is open.

For any open $U \subseteq X$, the set $i(U) = V \cap i(X)$ for some open $V \subseteq Y$, since $i$ is a homeomorphism onto its image. Since $i(X)$ is open, the map $i : X \to Y$ is open too. □

Proposition 3.43. Let $X$ be a locally compact, Hausdorff, non-compact space and suppose $i : X \to Y$ is a compactification where $Y$ is Hausdorff. Then the map $f : Y \to X \cup \{\infty\}$ given by

$$f(y) = \begin{cases} y & \text{if } y \in X \\ \infty & \text{otherwise} \end{cases}$$

is continuous.
This means that $X \cup \{\infty\}$ is final among all Hausdorff compactifications of $X$.

**Proof.** Let $U \subseteq X \cup \{\infty\}$ be an open set. We want to show that $f^{-1}(U)$ is open. There are two cases to consider.

First consider the case where $\infty \notin U$. In this case, the argument is clear enough: $U \subseteq X$ is open, so $i(U) = f^{-1}(U)$ is open by Lemma 3.42.

Second, consider the case when $\infty \in U$. Then $X \cup \{\infty\} \setminus U$ is a compact subset $K$ of $X$, and $i(K) = f^{-1}(X \cup \{\infty\} \setminus U)$ is closed in $Y$. The result follows. \qed

**Definition 3.44.** A commutative diagram of the form

$$
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow{f} & & \downarrow{f'} \\
Y' & \xrightarrow{i'} & Y
\end{array}
$$

where $i : X \to Y$ and $i' : X \to Y'$ are compactifications and $f$ is a continuous map is called a map of compactifications of $X$. A homeomorphism of compactifications is a map of compactifications with an inverse map, or equivalently, a map of compactifications where $f$ is a homeomorphism.

**Corollary 3.45.** Let $X$ be a locally compact, Hausdorff but not compact space, and let $i : X \to Y$ be a Hausdorff compactification in which $Y \setminus i(X)$ consists of one point. Then $i$ is homeomorphic to the one-point compactification.

**Proof.** There exists a continuous bijection $f : Y \to X \cup \{\infty\}$ in which the source is compact and the target is Hausdorff. \qed

**Example 3.46.** Using stereographic projection, we can prove that $S^n$ is homeomorphic to the one-point compactification of $\mathbb{R}^n$. This is on the homework.

**Remark 3.47.** A Hausdorff compactification of a locally compact Hausdorff space, $i : X \to Y$, is determined up to homeomorphism by which continuous functions $f : X \to Z$ where $Z$ is compact Hausdorff can be extended

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow{i} & & \downarrow{i'} \\
Y & \xrightarrow{i'} & Y
\end{array}
$$

Sometimes we can quantify over a smaller class than “all compact Hausdorff spaces”: For instance, the compactification $S^1$ of $\mathbb{R}$ has the property that a function $f : \mathbb{R} \to \mathbb{R}$
3.6. Compactly generated topologies

[0, 1] extends to a function \( \hat{f} : S^1 \to \mathbb{R} \) if and only if \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \) exist and agree.

There is a “two-point compactification of \( \mathbb{R} \)”, denoted \( \mathbb{R} \cup \{ \pm \infty \} \). This is actually homeomorphic to \([0, 1]\). A function \( f : \mathbb{R} \to [0, 1] \) extends to \( \hat{f} : \mathbb{R} \cup \{ \pm \infty \} \to [0, 1] \) if and only if \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \) exist, but they do not have to agree.

One might hope for a “maximal” Hausdorff compactification of a locally compact Hausdorff space \( X \), and this does exist (assuming the axiom of choice). It is called the \( \text{Stone–Čech compactification} \) \( i_X : X \to \beta X \). It has the property that any continuous function \( f : X \to Z \) where \( Z \) is compact Hausdorff has a unique extension to \( \beta f : \beta X \to Z \).

The one-point compactification allows us to prove the following proposition extending the characterization of locally compact Hausdorff spaces. It extends \ref{prop:locally_compact_characterization}.

**Proposition 3.48.** Let \( X \) be a Hausdorff topological space. Then \( X \) is locally compact if and only if, for all open \( N \ni x \), there exists an open \( U \ni x \) such that \( \bar{U} \subseteq N \) and \( \bar{U} \) is compact.

**Proof.** Suppose \( X \) is locally compact. Let \( X \cup \{ \infty \} \) denote the one-point compactification. This is a compact Hausdorff space, and is therefore normal. Consider the two closed sets in \( X \cup \{ \infty \} \):

\[(X \cup \{ \infty \}) \setminus N, \quad \{ x \}.
\]

These are disjoint, and by normality (regularity is enough), we can find disjoint open sets \( U, V \) containing them. The set \( U \) is an open set containing \( \infty \), so its complement is a compact subset \( K \subseteq X \), and \( U \) was constructed to contain \( X \setminus N \), so that \( K \subseteq N \). On the other hand, \( V \) is an open set not containing \( \infty \), and therefore \( V \subseteq X \) and \( V \) is an open set in \( X \). Moreover \( V \subseteq K \), since \( V \) and \( U \) are disjoint.

But then \( \bar{V} \) in \( X \) is a closed subset contained in \( K \), which is compact, so \( \bar{V} \) is compact and \( \bar{V} \subseteq N \).

The other direction is trivial. \( \square \)

### 3.6 Compactly generated topologies

**Definition 3.49.** Let \( X \) be a topological space. We say a subspace \( C \subseteq X \) is \( k \)-closed if \( u^{-1}(C) \) is closed in \( K \) for all continuous maps \( u : K \to X \) with compact Hausdorff source.

**Proposition 3.50.** The \( k \)-closed subsets form the closed sets of a topology on \( X \). If the set \( X \) equipped with this topology is denoted \( kX \), then the identity map \( kX \to X \) is continuous.

**Definition 3.51.** We say a topology on \( X \) is *compactly generated* if every \( k \)-closed set is closed.
Proposition 3.52. If $X$ is a topological space satisfying either of the following conditions, then $X$ is compactly generated:

1. $X$ is Hausdorff and locally compact.

2. Every sequentially closed subset of $X$ is closed (e.g. if $X$ is first countable).

Proof. 1. Suppose $C$ is $k$-closed and $x \in \overline{C}$. Let $K$ be a compact neighbourhood of $x$ (a compact set containing an open neighbourhood). Suppose $V \ni x$ is an arbitrary open neighbourhood, then $K \cap V$ contains an open neighbourhood of $x$ and so $K \cap V \cap C \neq \emptyset$. Therefore $x \in \overline{K \cap C}$. Let $j : K \to X$ be the inclusion. Then $j^{-1}(C) = K \cap C$ is closed in $K$, which implies that $x \in C$.

2. Exercise. □
3.6. Compactly generated topologies
Chapter 4

Connectedness

4.1 Connectedness

Definition 4.1. We say a topological space $X$ is connected if every function $X \rightarrow D$ where $D$ has a discrete topology is constant.

Lemma 4.2. Let $X$ be a topological space. The following are equivalent:

1. $X$ is connected;

2. Every function $X \rightarrow \{0, 1\}$, where the target is discrete, is constant.

3. If $A \subseteq X$ is open and closed, then $A = \emptyset$ or $A = X$.

Example 4.3. Let $A \subseteq \mathbb{R}$ be a subset of the real line. If $x < y < z$ are three points in $\mathbb{R}$ such that $x, z \in A$ but $y \not\in A$, then the function $f : \mathbb{R} \setminus \{y\} \rightarrow \{0, 1\}$ given by $f(t) = 0$ if $t < y$ and $f(t) = 1$ otherwise is continuous and so restricts to a continuous nonconstant function on $A$. So $A$ is not connected (it is disconnected).

This implies that if $A$ is a connected subset of $\mathbb{R}$, then $A$ is an interval (including the degenerate intervals $\emptyset$ and $\{a\}$). On the other hand, if $A$ is an interval and $f : A \rightarrow \mathbb{R}$ has the property that $f(x) = 0$ and $f(z) = 1$ (without loss of generality $x < z$) and that $f$ is continuous, then consider the element $y = \inf \{f^{-1}(1) \cap [x, z]\}$. Since $f^{-1}(1)$ is closed, $f(y) = 1$, but since $f^{-1}(0)$ is closed, $f(y) = 0$, a contradiction.

It follows that the connected subsets of $\mathbb{R}$ are precisely the intervals.

Proposition 4.4. Let $f : X \rightarrow Y$ be a continuous surjective function, and suppose $X$ is connected. Then $Y$ is connected.
Proposition 4.5. Let \( X \) be a topological space, let \( \{ A_i \}_{i \in I} \) be a family of connected subspaces of \( X \) such that for all \( i, j \in I \), the set \( A_i \cap A_j \) is nonempty and such that \( \bigcup_{i \in I} A_i = X \). Then \( X \) is connected.

Corollary 4.6. Let \( \{ X_i \}_{i \in I} \) be a family of connected spaces. Then \( X = \prod_{i \in I} X_i \) is connected.

Proof. If any of the sets \( X_i \) is empty, then the product is empty and there is nothing to do.

Let \( f : X \to \{0, 1\} \) be a continuous function that is not identically 0. First we observe that if \( y \) and \( y' \) are two points of \( X \) that differ only in one coordinate, the \( j \)-th coordinate, then \( f(y) = f(y') \). This is because there is an inclusion \( c_j : X_j \to \prod_{i \in I} X_i \) given by the other coordinates, and the composite \( f \circ c_j \) gives a continuous function \( X_j \to \{0, 1\} \).

By an easy induction, if \( y \) and \( y'' \) differ in finitely many coordinates, then \( f(y) = f(y'') \).

Finally, suppose \( x \in X \) is such that \( f(x) = 1 \). Then \( f^{-1}(1) \) contains a subbasic open set \( U \) around \( x \), so that there are finitely many coordinates \( i_1, \ldots, i_r \) such that if \( x \) and \( y'' \) agree in these coordinates, then \( y'' \in U \). In particular, \( f(y'') = 1 \) as well. But then for arbitrary \( y \in X \), we can change finitely many coordinates to produce \( y'' \in U \), so \( f(y) = 1 \). Therefore \( f \) is identically 1.

Definition 4.7. Let \( X \) be a topological space and let \( x \in X \) be point. The connected component \( C_x \) of \( x \) is the union of all connected subsets of \( X \) containing \( x \).

Remark 4.8. The connected components are connected, and if \( y \in C_x \), then \( C_y = C_x \).

Example 4.9. The space \( Q \) shows that the connected components are not necessarily open and closed. A space in which connected components are singletons is said to be totally disconnected.

The space \( Q \) seems badly behaved from a certain point of view: one might like the connected components themselves to be both closed and open subsets of the space, but this is not the case.

Remark 4.10. Every topological space is a union of connected components, and these components are pairwise disjoint.

Definition 4.11. A space \( X \) is locally connected if every point \( x \in X \) has a local base \( \{ U_i \}_{i \in I} \) such that \( U_i \) is connected.

Proposition 4.12. Let \( X \) be a locally connected space, and let \( X_\alpha \) be the connected component of \( x \in X \). Then \( X_\alpha \) is open and closed.
Proof. It suffices to prove $X_\alpha$ is open, since then the complement is the union of the other components of $X$, which is also open.

The component $X_\alpha$ contains open neighbourhoods around each point, and is therefore open. □

4.2 Path-connectedness

Let $I$ denote $[0, 1]$ with the usual topology throughout.

Definition 4.13. A topological space $X$ is path-connected if for every two points $x, y \in X$, there is a continuous function $\gamma : I \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 4.14. If $X$ is a topological space and $\{A_i\}$ is a set of path connected subsets that cover $X$ and such that $\bigcap A_i \neq \emptyset$, then $X$ is path connected.

Corollary 4.15. If $X$ is path connected, then it is connected.

Proposition 4.16. If $\{X_j\}_{j \in J}$ is a family of path-connected topological spaces, then $X = \prod_{j \in J} X_j$ is path-connected.

Definition 4.17. A path component of $X$ is a maximal path-connected subspace. Write $P_x$ for the path component of $x \in X$.

Remark 4.18. As in the case of connected components, $y \in P_x$ if and only if $x \in P_y$.

Definition 4.19. A space $X$ is locally path connected if every point has a local base consisting of path connected neighbourhoods.

Proposition 4.20. Suppose $X$ is a connected locally path-connected space. Then $X$ is path-connected.

Proof. As in the case of connectedness, the locally path-connected hypothesis implies that path-components are open and closed. The space $X$ can have no nontrivial open-and-closed sets, so $X$ has a single path-component. □

Example 4.21. The topologist’s sine curve $S$ is a well-known metric space that is connected but not path connected. This set is defined as

$$S = \{(0, y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin \frac{1}{x}) \mid x > 0\}.$$ 

First we show that this is connected. The subset $C = \{(x, \sin \frac{1}{x}) \mid x > 0\}$ is homeomorphic to $(0, \infty)$ and is therefore connected. Let $(0, t)$ be a point in $L = \{(0, y) \mid -1 \leq y \leq 1\}$.
4.2. Path-connectedness

1), and let $f : S \to \{0, 1\}$ be a continuous function taking (without loss of generality) the value 0 on $C$. Then there is a sequence in $C$ converging to $(0, t)$, so that $f((0, t)) = 0$ by continuity. The function $f$ is therefore constant. This shows $S$ is connected.

Now we show that $S$ is not path connected. Suppose $f : I \to S$ is a path from a point in $L$ to $p = (1/\pi, 0)$. Consider the set $f^{-1}(L)$, which is closed in $L$, and therefore has a maximal element, $t_0 < 1$. Restricting, we have a continuous function $f : [t_0, 1] \to S$ with the property that $f(t_0) = (0, y_{t_0})$ and $(x_t, y_t) := f(t) \in C$ for $t > t_0$. Since $f$ is continuous, $\lim_{t \to t_0^+} x_t = 0$, and by the intermediate value theorem we can find sequences of values of $t$

$$(t_n^+) \quad \text{such that } x_{t_n^+} = 1/(2n\pi)$$

and

$$(t_n^-) \quad \text{such that } x_{t_n^-} = 3/(2n\pi)$$

each converging to $t_0$. But then $f(t_n^+) = (t_n^+, 1)$ while $f(t_n^-) = (t_n^-, -1)$, so that the $y$ coordinate of $f(t_0)$ must be both $\lim_{n \to \infty} +1$ and $\lim_{n \to \infty} -1$. 

Chapter 5

Homotopy

5.1 The homotopy category

In this section, “map” will denote a continuous function and $I$ will denote $[0, 1]$.

**Definition 5.1.** Let $X$ and $Y$ be topological spaces and $A \subseteq X$ be a subset. Suppose $f_0, f_1 : X \to Y$ are two maps such that $f_0|_A = f_1|_A$. Then a homotopy relative to $A$ from $f_0$ to $f_1$ (or between $f_0$ and $f_1$) is a map $H : X \times I \to Y$ such that

- $H(x, 0) = f_0(x)$ $\forall x \in X$
- $H(x, 1) = f_1(x)$ $\forall x \in X$
- $H(a, t) = f_0(a) = f_1(a)$ $\forall a \in A$, $\forall t \in I$

**Notation 5.2.** A homotopy relative to $\emptyset$ is called a homotopy.

**Notation 5.3.** If a homotopy relative to $A$ exists from $f$ to $g$, we say that $f$ and $g$ are homotopic relative to $A$. In the case where $A = \emptyset$, we say $f$ and $g$ are homotopic.

**Proposition 5.4.** Let $X$ and $Y$ be topological spaces, and $A \subseteq X$ a subset. Write $f_0 \sim f_1$ if $f_0$ is homotopic to $f_1$ relative to $A$. Then $\sim$ is an equivalence relation.

**Proof.** Reflexivity: $H(x, t) = f(x)$ gives a homotopy from $f$ to $f$.

Symmetry: if $H$ gives a homotopy one way, then $H'(x, t) := H(x, 1 - t)$ gives a homotopy the other way.

Transitivity: suppose $H_0$ is a homotopy (rel. $A$) from $f_0$ to $f_1$ and $H_1$ is a homotopy (rel. $A$) from $f_1$ to $f_2$. Then define

$$H(x, t) = \begin{cases} H_0(x, 2t) & t \leq 1/2 \\ H_1(x, 2t - 1) & t > 1/2 \end{cases}$$
This gives a homotopy from $f_0$ to $f_2$ (rel. $A$).

**Notation 5.5.** We write $f \sim_A g$ to indicate that $f$ is homotopic to $g$ relative to $A$. This implies, among other things, that $f|_A = g|_A$.

**Proposition 5.6.** Suppose $X$, $Y$ and $Z$ are three spaces, and $A \subseteq X$ and $B \subseteq Y$ are subspaces. Suppose $f_0, f_1 : X \to Y$ are maps such that $f_0 \sim_A f_1$ and that $f_0(A) \subseteq B$. Suppose also that $g_0, g_1 : Y \to Z$ are maps such that $g_0 \sim_B g_1$. Then $g_1 \circ f_1 \sim_A g_0 \circ f_0$.

**Proof.** There exist homotopies $H'$ from $f_0$ to $f_1$ and $H''$ from $g_0$ to $g_1$. Now consider $H : X \times I \to Z$ defined by $H(x, t) = H''(H'(x, t), t)$. This is the required homotopy.

**Definition 5.7.** The homotopy category $\mathcal{H}$ is a category defined as follows. The objects of $\mathcal{H}$ are the topological spaces. The morphisms in $\mathcal{H}$ from $X$ to $Y$ are the homotopy classes (rel. $\emptyset$) of maps $X \to Y$.

**Notation 5.8.** In this course, we will write $[X, Y]$ for the set of morphisms $X \to Y$ in $\mathcal{H}$.

**Remark 5.9.** There is a functor $\text{Top} \to \mathcal{H}$ that is the identity on objects and sends a morphism $f : X \to Y$ to $[f]$, the homotopy class of $f$.

Once we have a category, the homotopy category, we can ask what the isomorphisms are.

**Definition 5.10.** A map $f : X \to Y$ of topological spaces is a homotopy equivalence if $[f]$ is an isomorphism in $\mathcal{H}$.

**Remark 5.11.** Unwinding this definition, $f : X \to Y$ is a homotopy equivalence if there exists an inverse morphism for $[f]$ in the homotopy category. That is, $f$ is a homotopy equivalence if there exists some $g : Y \to X$ such that $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$.

**Remark 5.12.** Since the homotopy equivalences are exactly the maps that become an isomorphism after application of the functor $[\cdot]$, they satisfy the two-out-of-three (and the three-out-of-six) property.

**Definition 5.13.** An embedding $i : A \to X$ is a retract if there exists a retraction map $r : X \to A$ such that $i \circ r(a) = a$ for all $a \in A$. 

---
Definition 5.14. An embedding \( i : A \to X \) is a (weak) deformation retract if there exists a retraction map \( r : X \to A \) such that the composite \( i \circ r : X \to X \) is homotopic to \( \text{id}_X \). If it is homotopic to \( \text{id}_X \) relative to \( A \), then we say it is a strong deformation retract. Most deformation retracts one encounters are strong deformation retracts.

Remark 5.15. As we often do, if \( i : A \to X \) is a retract or a deformation retract, then we will simply say “\( A \) is a (deformation) retract of \( X \)” if \( i \) is obvious, for instance, if \( A \) is a subspace of \( X \) so \( i \) is the inclusion.

Lemma 5.16. If \( i : A \to X \) is a weak deformation retract, then \( i \) is a homotopy equivalence.

Proof. We know \( i \circ r \simeq \text{id}_X \) and \( r \circ i = \text{id}_A \), which establishes the homotopy equivalence. \( \square \)

Example 5.17. If \( X \) is any nonempty space at all, then any inclusion \( \{x\} \to X \) is a retract, but in general this is not a deformation retract.

Example 5.18. Recall that \( S^n \) denotes the subset of \( \mathbb{R}^{n+1} \) consisting of elements of norm 1. This includes as a subspace \( i : S^n \to \mathbb{R}^{n+1} \setminus \{0\} \). There is a function \( r : \mathbb{R}^{n+1} \setminus \{0\} \to S^n \) given by \( r(\vec{v}) = \frac{\vec{v}}{||\vec{v}||} \), which is clearly a retraction and is actually a strong deformation retraction via the homotopy
\[
(\vec{v}, t) \mapsto \frac{1}{t(||\vec{v}|| - 1)} + 1 \vec{v}.
\]

5.2 The pointed homotopy category and homotopy groups

Definition 5.19. Let the category \( \text{Top}_\bullet \) denote the category of based topological spaces. The objects are pairs \( (X, x_0) \) where \( x_0 \in X \), and the morphisms \( f : (X, x_0) \to (Y, y_0) \) are continuous functions such that \( f(x_0) = y_0 \).

Remark 5.20. We frequently omit the basepoint \( x_0 \) from the notation.

Notation 5.21. Write \( [(X, x_0), (Y, y_0)]_\bullet \) for the set of homotopy classes relative to \( x_0 \), of maps \( f : X \to Y \) satisfying \( f(x_0) = y_0 \). That is, \( g \) is in the class of \( f \) if and only if there exists a homotopy
\[
H : X \times I \to Y
\]
such that
\[
H(x, 0) = f(x) \quad \forall x \in X
\]
\[
H(x, 1) = g(x) \quad \forall x \in X
\]
\[
H(x_0, t) = y_0 \quad \forall t \in I
\]
Definition 5.22. We write $\mathbf{H}_\bullet$ for the **pointed homotopy category**: the objects are based spaces and the set of morphisms from $X$ to $Y$ is $[(X, x_0), (Y, y_0)]_\bullet$.

Remark 5.23. As in the unpointed case, there is a functor $\mathbf{Top}_\bullet \to \mathbf{H}_\bullet$ that is the identity on objects and sends $f$ to the (pointed) homotopy class of $f$.

Remark 5.24. There can exist based spaces $(X, x_0)$ and $(Y, y_0)$, along with pointed maps $f, g : X \to Y$ such that $f \simeq g$ in the unpointed sense, but $f \not\simeq_{x_0} g$ in the pointed sense.

Definition 5.25. Let $(X, x_0)$ be a pointed space and let $n \geq 0$. Give $S^n$ the basepoint $(1, 0, \ldots, 0)$. Then define the $n$-th homotopy group of $(X, x_0)$ by

$$\pi_n(X, x_0) = [S^n, X]_\bullet$$

Note that the basepoint is dropped from the notation.

Remark 5.26. If you prefer not to use the homotopy-category formulation, you can define $\pi_n(X, x_0)$ as follows:

$$\pi_n(X, x_0) = \{f : (S^n, s_0) \to (X, x_0) \mid f \text{ is a pointed map}\}/\simeq$$

where the relation $\simeq$ denotes pointed homotopy.

Remark 5.27. The term “homotopy group” is a misnomer, since $\pi_0(X, x_0) = [S^0, X]_\bullet$ is not in general a group. For larger values of $n$, however, the sets $\pi_n(X, x_0)$ do have a natural group operation defined on them. We will prove this in the case of $\pi_1(X, x_0)$ later.

Example 5.28. $\pi_0(X, x_0)$ is the set of based homotopy classes of based maps $S^0 \to (X, x_0)$. The space $S^0$ is the discrete space with two points. A based map $f : S^0 \to X$ corresponds exactly to a single element $f(-1) \in X$, since $f(1) = x_0$. Then two such maps $f, g$ are homotopic if and only if there is a path in $X$ from $f(-1)$ to $g(-1)$. Therefore we can identify $\pi_0(X, x_0)$ with the set of path components of $X$. There is a distinguished component, that of $x_0$.

In contrast to the case of the larger values of $i$, in the case where $i = 0$, one can define $\pi_0(X)$ without reference to the basepoint: $\pi_0(X) = [\ast, X]$. This is the set of path components of $X$ without any distinguished choice of component.

Remark 5.29. As defined, $\pi_n(X, x_0)$ is a composite of two functors: one being the functor from $\mathbf{Top}_\bullet \to \mathbf{H}_\bullet$ and the other being $[S^n, \ast]_\bullet$. This implies that $\pi_n(X, x_0)$ is itself a functor, or, in less fancy language, if you have a based map $f : (X, x_0) \to (Y, y_0)$, then you get induced functions

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$
and these induced functions respect compositions and identities.

To define \( f_* \) explicitly with functions, do the following: suppose \( g : (S^n, s_0) \to (X, x_0) \) is a map of pointed spaces. Then \( g \) represents a class \([g] \in \pi_n(X, x_0)\). The element \( f_*([g]) \) is the homotopy class of the map \( f \circ g : (S^n, s_0) \to (Y, y_0)\).

**Remark 5.30.** If \( h : (X, x_0) \to (Y, y_0) \) is an isomorphism in \( H_* \), then \([S^n, X]_* \to [S^n, Y]_*\) is an isomorphism. That is, if \( h \) is a pointed homotopy equivalence, then the induced map \( h_* : \pi_n(X, x_0) \to \pi_n(Y, y_0) \) is a bijection.

This can be strengthened to the statement: If \( h : X \to Y \) is an isomorphism in \( H_\ast \), and \( x_0 \in X \), then the induced map \( h_* : \pi_n(X, x_0) \to \pi_n(Y, h(y_0)) \) is a bijection. We will not prove this for \( n \geq 2 \) in this course. For \( n = 0 \), it is easy to see because \( \pi_0(X, x_0) \) is the set \( \pi_0(X) = [*, X] \) equipped with a distinguished point, and \( \pi_0(\cdot) \) does not depend on the basepoint. The result for \( \pi_1 \) will be proved later.

**Remark 5.31.** The groups \( \pi_i(X, x_0) \) are considered hard to compute when \( i \geq 2 \). For instance, I don’t think \( \pi_i(S^2, s_0) \) has been fully computed for values of \( i \) much beyond 30. We will concentrate in the rest of this course on the group about which a lot is known: \( \pi_1(X, x_0) \). This is the fundamental group of \( (X, x_0) \).

**Proposition 5.32.** Let \( \{X_j, x_j\}_{j \in J} \) be a family of based topological spaces. Let \( (X, x) \) denote the product, based at the point which projects onto \( x_j \) for all \( j \). Let \( n \geq 0 \). Then the natural map \( \pi_n(X, x) \to \prod_{j \in J} \pi_n(X_j, x_j) \) is an isomorphism.

**Proof.** In fact, the universal property of products says that a family of pointed maps \( \{g_i : (S^n, s_0) \to (X, x_0)\}_{i \in I} \) is equivalent to a single pointed map \( \prod_{i \in I} g_i : (S^n, s_0) \to (\prod_{i \in I} X, (x_i)) \), and similarly, a family of homotopies \( \{H_i : (S^n, s_0) \times I \to (X, x_0)\}_{i \in I} \) is equivalent to a single homotopy \( \prod_{i \in I} H_i : (S^n, s_0) \times I \to (\prod_{i \in I} X, (x_i)) \). This is enough to establish that the map in the proposition is a bijection. \( \square \)

**Construction 5.33.** Let \( (X, x_0) \) and \( (Y, y_0) \) be two pointed spaces. We define the wedge sum \( (X, x_0) \lor (Y, y_0) \) to be the quotient of \( X \coprod Y \) given by collapsing the subspace \( \{x_0\} \coprod \{y_0\} \). The space \( (X, x_0) \lor (Y, y_0) \) is naturally embedded as a subspace of \( X \times Y \), by equating \( x \in X \) with the pair \((x, y_0)\) and \( y \in Y \) with \((x_0, y)\). The quotient \((X \times Y)/(X \lor Y) =: X \land Y\) is called the smash product.

If \( X \) is the one-point compactification of an LCH space \( U \) and \( Y \) is the one-point compactification of an LCH space \( V \), and if \( X \) and \( Y \) are given the basepoints at \( \infty \), then \( X \land Y \) is the one-point compactification of \( U \times V \).

In particular, \( S^n \land S^m \approx S^{n+m} \).
5.3 Contractible spaces

Definition 5.34. A map \( f : X \to Y \) is nullhomotopic if there exists an element \( y \in Y \) such that \( f \) is homotopic to the map with constant value \( y \). Similarly, a pointed map \( f : (X, x_0) \to (Y, y_0) \) is nullhomotopic if it is homotopy equivalent (in the pointed sense) to the constant map, which must be the map sending \( X \) to \( y_0 \in Y \). If a map is not nullhomotopic, it is essential.

Remark 5.35. If a pointed map \( f : (X, x_0) \to (Y, y_0) \) is nullhomotopic, then the underlying (unpointed) map \( f \) is nullhomotopic. In practice, it is rare that we encounter a pointed map \( f : (X, x_0) \to (Y, y_0) \) so that the underlying map \( f \) is nullhomotopic but where the pointed map is essential. An example is given by an inclusion of a point in the comb-space, so they do exist.

Definition 5.36. A space \( X \) is contractible if the identity map \( \text{id}_X : X \to X \) is nullhomotopic.

Remark 5.37. There exists an analogous condition of pointed contractibility: a pointed space \( (X, x_0) \) is pointed-contractible if the identity \( \text{id} : (X, x_0) \to (X, x_0) \) is homotopic to the constant map. Again, the comb-space can be used to produce a pointed space \( (X, x_0) \) so that \( (X, x_0) \) is not pointed-contractible, but \( X \) is contractible (forgetting the basepoint). Even worse, there exist examples of spaces \( Y \) that are contractible but such that \( (Y, y_0) \) is not pointed-contractible for any choice of \( y_0 \in Y \).

For the classes of space that arise in practice, e.g., Manifolds or CW complexes (see later), one can often prove a result such as: \( (X, x_0) \) is pointed contractible if and only if \( X \) is contractible. Furthermore, it is the concept of contractibility, not pointed contractibility, that is most useful in applications. We do not really pursue pointed contractibility further in these notes.

Proposition 5.38. A space \( X \) is contractible if and only if \( X \) is homotopy equivalent to a one-point space, \( * \).

Proof. Suppose \( X \) is contractible, there exists a point \( x_0 \in X \) such that \( \text{id}_X \simeq f \) where \( f : X \to X \) is the constant map at \( x_0 \). We can factor the identity morphism \([\text{id}_X] : X \to X\) in \( \mathbf{H} \) as \([\text{id}_X] = [x_0 \to X] \circ [X \to *]\). In the other order, \([X \to *] \circ [x_0 \to X] = [\text{id}_*]\). Therefore \([X \to *]\) is a homotopy equivalence.

Conversely, suppose the unique morphism \( X \to * \) is a homotopy equivalence. Let \( f : * \to X \) be a homotopy inverse for this map. Then the identity \( \text{id}_X : X \to X \) is homotopic to the constant map with value \( f(*) \). \( \Box \)


**Proposition 5.39.** Suppose $X$ is a contractible space and $Y$ is any space. The set $[Y, X]$ of homotopy classes of maps from $Y$ to $X$ consists of one element.

That is to say, every map from $Y$ to $X$ is homotopic to every other map.

*Proof.* Since $X$ is contractible, we can find an element $x_0 \in X$ so that $\text{id}_X : X \to X$ is homotopic to $g : X \to X$, the constant map with value $x_0$. Then if $f : Y \to X$ is any map, there is a homotopy from $f = \text{id}_X \circ f$ to $g \circ f$, and $g \circ f$ is the map $Y \to X$ with the constant value $x_0$.

**Corollary 5.40.** If $X$ is contractible and $i : A \to X$ is a retract, then $A$ is contractible.

(Note: the hypothesis says “retract” and not “deformation retract”)

*Proof.* We show that if $i : A \to X$ is a retract, then $A$ is homotopy equivalent to $X$, and therefore to a point. Let $r : X \to A$ be a retraction, so that $r \circ i = \text{id}_A$. But $i \circ r$ is a map from $X \to X$, and is therefore homotopic to $\text{id}_X$ (since all maps $X \to X$ are homotopic by Proposition 5.39).

In summary, $r \circ i = \text{id}_A$ and $i \circ r \simeq \text{id}_X$, so that $A \simeq X$. Since $X$ is contractible, so is $A$.

Many commonly-occurring spaces are contractible. Here is a device for establishing this quickly.

**Proposition 5.41.** Suppose a space $X$ has a multiplicative action by $[0,1]$; that is, there is a map $m : [0,1] \times X \to X$ for which

1. $m(t_1, m(t_2, x)) = m(t_1 t_2, x)$ and
2. $m(1, x) = x$

for all $t_1, t_2 \in [0,1]$ and $x \in X$. Let $X_0$ denote the image of $m(0, \cdot)$. Then $X_0$ is a (strong) deformation retract of $X$.

*Proof.* Certainly $i : X_0 \to X$ is an embedding. The map $m : [0,1] \times X \to X$ functions as the homotopy in a strong deformation retract.

**Corollary 5.42.** Let $V$ be a topological R-vector space, and let $X$ be a nonempty subspace of $V$ that is closed under multiplication by $[0,1]$ (call such a region starshaped). Then $X$ is contractible.

*Remark 5.43.* If $V$ is as above and $Y$ is a nonempty convex subset of $V$, then $Y$ is homeomorphic to a translate of $Y$ containing the origin. This translate is starshaped, and therefore $Y$ is contractible.
5.3. Contractible spaces

Initial and terminal objects

Remark 5.44. In a category $\mathbf{C}$, if an object $\ast$ has the property that all objects $c \in \mathbf{C}$ are have a unique map $c \to \ast$, then $\ast$ is called a terminal object of $\mathbf{C}$. You can prove many trivial things about terminal objects: for instance, any two terminal objects must be isomorphic and any map between terminal objects must be an isomorphism. Any object isomorphic to a terminal object must itself be terminal.

A space $X$ is contractible if and only if $X$ is a terminal object of $\mathbf{H}$. The trivialities mentioned above lead to abstract proofs of statements such as: any two maps $f, g : X \to Y$ between contractible spaces are homotopic.

Remark 5.45. There is also the notion of an initial object $\emptyset$ in a category $\mathbf{C}$. An object is initial if all objects $c$ are equipped with a unique morphism $\emptyset \to c$. Similarly to the case of final objects, initial objects are isomorphic, all maps between them are isomorphisms and all objects isomorphic to an initial object must be initial.

There is a unique initial object in $\mathbf{Top}$ and $\mathbf{H}$, the empty space. In $\mathbf{Top}_*$ and $\mathbf{H}_*$, the one-point spaces are initial, however. An object that is both initial and final is called a zero object.
Chapter 6

The fundamental groupoid and the fundamental group

6.1 The fundamental groupoid

Construction 6.1. Suppose \( \gamma, \delta : I \to X \) are two paths, and suppose \( \gamma(1) = \delta(0) \). We define a composite path \( \gamma \cdot \delta : I \to X \) by

\[
\gamma \cdot \delta(t) = \begin{cases} 
\gamma(2t) & \text{if } t \leq 1/2 \\
\delta(2t - 1) & \text{if } t \geq 1/2 
\end{cases}
\]

Notation 6.2. Let us say that two paths \( \gamma, \gamma' : I \to X \) are equivalent and if \( \gamma \simeq \gamma' \) relative to \( \{0, 1\} \). In particular, \( \gamma \) and \( \gamma' \) have the same endpoints. Recall that homotopy (relative to a subspace) is an equivalence relation. We will write \([\gamma]\) for the equivalence class of \( \gamma \).

Remark 6.3. Warning: the notation \([\gamma]\) when \( \gamma : I \to X \) is different from the notation \([f]\) used in the previous chapter. The homotopies used to define \([\gamma]\) are the homotopies relative to \( \{0, 1\} \), and when \( \gamma : I \to X \) is a path, the notation \([\gamma]\) will be used only in this sense.

What follows is some technical lemmas about path composition.

Proposition 6.4. If \([\gamma] = [\gamma']\) and \([\delta] = [\delta']\) and if \( \gamma \cdot \delta \) is defined, then \([\gamma \cdot \delta] = [\gamma' \cdot \delta']\).

Remark 6.5. That is, the composition of paths descends to equivalence classes. After we have proved this result, we can define \([\gamma][\delta]\) by choosing representatives.
6.1. The fundamental groupoid

**Proof.** Let \( H \) be a homotopy from \( \gamma \) to \( \gamma' \), relative to endpoints, and similarly, let \( E \) be a homotopy from \( \delta \) to \( \delta' \). Then define

\[
H \cdot E : I \times I \to X, \quad E \circ H(t, s) = \begin{cases} H(2t, s) & \text{if } t \leq 1/2 \\ E(2t - 1, s) & \text{if } t \geq 1/2 \end{cases}
\]

This gives the required homotopy of \( \gamma \cdot \delta \) to \( \gamma' \cdot \delta' \) relative to \( \{0, 1\} \). \( \square \)

Composition of paths is not associative—you can check directly that \( \gamma \cdot (\delta \cdot \epsilon) \neq (\gamma \cdot \delta) \cdot \epsilon \).

**Proposition 6.6.** Suppose \( \gamma, \delta \) and \( \epsilon \) are paths in \( X \) such that \( \gamma \cdot (\delta \cdot \epsilon) \) is defined. Then \([\gamma][(\delta)[\epsilon]] = ([\gamma][\delta])[\epsilon]\).

That is, the composition is associative once we pass to homotopy classes.

**Proof.** It is sufficient to write down a homotopy (relative to endpoints) between \( \gamma \cdot (\delta \cdot \epsilon) \) and \( (\gamma \cdot \delta) \cdot \epsilon \).

\[
H(t, s) = \begin{cases} \gamma(2t + 2st) & \text{if } t \leq 1/2 - s/4 \\ \delta(4t - 2 + s) & \text{if } 1/2 - s/4 \leq t \leq 3/4 - s/4 \\ \epsilon(4t - 2st + 2s - 3) & \text{if } t \geq 3/4 - s/4 \end{cases}
\]

\( \square \)

**Definition 6.7.** If \( X \) is a space and \( x \in X \), define \( e_x \) to be the constant path at \( x \), i.e., \( e_x(t) = x \) for all \( t \).

**Proposition 6.8.** Let \( X \) be a space and let \( \gamma \) be a path in \( X \) starting at \( x \) and ending at \( y \). Then \([e_x] \cdot [\gamma] = [\gamma] \) and \([\gamma] \cdot [e_y] = [\gamma]\).

**Proof.** We’ll show one of these. The other is similar.

Just write down a homotopy from \( e_x \cdot \gamma \) to \( \gamma \).

\[
H(t, s) = \begin{cases} x & \text{if } 2t \leq 1 - s \\ \gamma(2t - st - 1 + s) & \text{if } 2t \geq 1 - s \end{cases}
\]

\( \square \)

**Notation 6.9.** If \( \gamma : I \to X \) is a path, write \( \gamma^\leftarrow \) for the reverse of \( \gamma \): \( \gamma^\leftarrow(t) = \gamma(1 - t) \). Clearly, \( (\gamma^\leftarrow)^\leftarrow = \gamma \).
Proposition 6.10. In the notation above, if \( \gamma \) is a path from \( x \) to \( y \), then \( [\gamma] \cdot [\gamma^{-}] = [e_{x}] \).

Proof. We write down a homotopy:

\[
H(t, s) = \begin{cases} 
\gamma(t) & \text{if } 2t \leq 1 - s \\
\gamma(1 - s) & \text{if } 1 - s \leq 2t \leq 1 + s \\
\gamma(2 - 2t) & \text{if } 1 + s \leq 2t
\end{cases}
\]

\[\square\]

Definition 6.11. A groupoid \( \mathcal{G} \) is a category having a set of objects and a set of morphisms and such that all morphisms are isomorphisms.

Definition 6.12. Let \( X \) be a topological space and let \( A \subseteq X \) be a subset of \( X \). Define a fundamental groupoid of \( X \) with endpoints in \( A \), denoted \( \Pi(X, A) \), as the groupoid where

\[
\text{ob} \, \Pi(X, A) = A
\]

and for \( a_{0}, a_{1} \in A \), the set of morphisms from \( a_{0} \) to \( a_{1} \), is the equivalence classes of paths in \( X \) starting at \( a_{0} \) and ending at \( a_{1} \). The previous propositions ensure that the composition law is well defined \(^1\) and associative, that \( [e_{a_{0}}] \) is the identity at \( a_{0} \) and that inverses exist for all morphisms (just reverse the path). So this really is a groupoid.

Remark 6.13. A special cases of the above is when \( A = X \). In this case \( \Pi(X, X) \) is often written simply \( \Pi(X) \), and is called the fundamental groupoid of \( X \).

6.2 More about groupoids

Notation 6.14. If \( a, b \) are two objects in a groupoid \( \mathcal{G} \), we will write \( \mathcal{G}(a, b) \) to denote the set \( \text{Mor}_{\mathcal{G}}(a, b) \).

Example 6.15. If a groupoid \( \mathcal{G} \) has a unique object *, then the data of the groupoid is really just \( \mathcal{G}(*, *) = \text{Mor}_{\mathcal{G}}(*, *) \). This is a set equipped with an associative composition law and an identity element, and where each element has an inverse. We call it a group.

Example 6.16. You should think of a groupoid \( \mathcal{G} \) as an algebraic thing, rather like a group. The starring role is played by the morphisms in \( \mathcal{G} \). The main difference between groups and groupoids is that if \( g_{1}, g_{2} \) are morphisms in a groupoid \( \mathcal{G} \), their composition \( g_{2} \circ g_{1} \) may not be defined.

\(^{1}\) Warning: one defines the geometric composition of paths \( [\gamma_{1}] \cdot [\gamma_{2}] \) to mean “first do \( \gamma_{1} \) and then do \( \gamma_{2} \). In order to match the conventions for compositions in a category, you have to declare that \( [\gamma_{2}] \circ [\gamma_{1}] = [\gamma_{1}] \cdot [\gamma_{2}] \). We use the geometric presentation throughout.
Remark 6.17. In the case of a point \( x_0 \) in a space \( X \), the groupoid \( \Pi(X, \{x_0\}) \) consists of homotopy classes (relative to \( \{0, 1\} \)) of maps \( \gamma : I \to X \), starting and ending at \( x_0 \). This is the same as homotopy classes of maps \( \gamma : S^1 \to X \) (since \( S^1 \approx I/\{0, 1\} \)), relative to the basepoint of \( S^1 \), which is \( [S^1, X]_* = \pi_1(X, x_0) \).

So we identify the groupoid \( \Pi(X, \{x_0\}) \) with \( \pi_1(X, x_0) \). This is called the fundamental group of \( X \) with basepoint \( x_0 \), and it is indeed a group, since it is a groupoid with one object.

Definition 6.18. If \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are two groupoids, then a morphism of groupoids \( \phi : \mathcal{G}_1 \to \mathcal{G}_2 \) is a functor. This definition is equivalent to saying that \( \phi \) consists of two functions

1. a function \( \phi \) from the objects of \( \mathcal{G}_1 \) to the objects of \( \mathcal{G}_2 \);
2. a function (also denoted \( \phi \)) from the morphisms of \( \mathcal{G}_1 \) to the morphisms of \( \mathcal{G}_2 \).

and these satisfy the conditions

1. if \( f : a \to b \) is an arrow in \( \mathcal{G}_1 \), then \( \phi(f) : \phi(a) \to \phi(b) \) is an arrow in \( \mathcal{G}_2 \);
2. if \( g \circ f \) is a composition in \( \mathcal{G}_1 \), then \( \phi(g \circ f) = \phi(g) \circ \phi(f) \).

Remark 6.19. You can verify that if \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are actually groups, this recovers the definition of homomorphism of groups.

Remark 6.20. There is a category of groupoids: the objects are the groupoids and the morphisms are the morphisms of groupoids.

Proposition 6.21. Suppose \( \mathcal{G} \) is a groupoid and \( f : a \to b \) is an arrow in \( \mathcal{G} \). Then \( \mathcal{G}(a, a) \) is isomorphic to \( \mathcal{G}(b, b) \).

Proof. We may define a function \( \phi_f : \mathcal{G}(a, a) \to \mathcal{G}(b, b) \) by \( g \mapsto f \circ g \circ f^{-1} \). It is easy to verify that this is actually a homomorphism of groups, and that it has an inverse map \( \phi_{f^{-1}} \). \( \square \)

Notation 6.22. Say that a groupoid \( \mathcal{G} \) is connected if for all \( a, b \in \text{ob} \; \mathcal{G} \), the set \( \mathcal{G}(a, b) \) is not empty. We saw that in a connected groupoid that all the groups \( \mathcal{G}(a, a) \) are isomorphic. In a disconnected groupoid, that need not be the case.
6.3 Fundamental groupoids of spaces again

Remark 6.23. If \( f : X \to Y \) is a continuous map of spaces, and if \( A \subseteq B \), then there is an induced map of groupoids \( f_* : \Pi(X, A) \to \Pi(Y, f(A)) \), given by sending a point \( a \) to \( f(a) \) and the class of a path \( \gamma : I \to X \) to the class of \( f \circ \gamma \). We have already established enough about relative homotopy and maps to deduce that \( f_* \) is well defined and that \( (f \circ g)_* = f_* \circ g_* \) whenever \( f, g \) are composable maps, and that \( (\text{id}_X)_* \) is the identity map on groupoids.

Remark 6.24. The term pair, unless otherwise indicated, means a topological space \( X \) and a subspace \( A \), denoted \((X, A)\). There is a category of pairs, \( \text{Pair} \), where the objects are the pairs and a morphism \( f : (X, A) \to (Y, B) \) is a continuous map \( f : X \to Y \) such that \( f(A) \subseteq B \).

We have constructed a functor

\[ \Pi : \text{Pair} \to \text{Groupoid} \]

Proposition 6.25. Suppose \( f, g : (X, A) \to (Y, B) \) are maps of pairs and that \( f \simeq g \) relative to \( A \). Then \( f_* = g_* : \Pi(X, A) \to \Pi(Y, B) \).

Remark 6.26. In order to write down what happens when the homotopy is not relative to \( A \), which is the more interesting case, we have to define the concept of a natural transformation. We may do this later, if there’s time.

6.4 Basepoints and the fundamental group

To avoid getting bogged down in category theory, we often restrict ourselves to the special case \( A = \{x_0\} \), so that \( \Pi(X, A) = \pi_1(X, x_0) \).

If we are going to work with this seriously, we have to pay attention to basepoints. The following are useful results.

Proposition 6.27. Let \( X \) be a topological space, let \( x_0 \) and \( x_1 \) be points in \( X \) and let \( \alpha \) be a path from \( x_0 \) to \( x_1 \). There is an isomorphism \( \phi_\alpha : \pi_1(X, x_0) \to \pi_1(X, x_1) \) given by \( \gamma \mapsto \alpha^{-1} \gamma \alpha \).

The proof of this is immediate if we consider \( \pi_1(X, x_0) \) as a subgroupoid of \( \Pi(X) \).

Proposition 6.28. Let \( X \) and \( Y \) be two spaces and let \( f, g : X \to Y \) be two maps and let \( H : X \times I \to Y \) be a homotopy between them (basepoint free). Let \( x_0 \in X \). Let \( \alpha \) be the path
6.5. The van Kampen theorem

Let $t \mapsto H(x_0, t)$, from $f(x_0)$ to $g(x_0)$, and let $\phi_\alpha$ be as above. Then the diagram

\[\begin{array}{ccc}
\pi_1(Y, f(x_0)) & \xrightarrow{\phi_\alpha} & \pi_1(Y, g(x_0)) \\
f_* & & g_* \\
\pi_1(X, x_0) & \xrightarrow{\cong} & \pi_1(Y, g(x_0))
\end{array}\]

commutes.

Proof. Let $\gamma$ be a loop in $X$ based at $x_0$. There are two ways of producing a loop in $Y$, based at $g(x_0)$. First, one can produce $(\alpha^{-1} \cdot (f \circ \gamma)) \cdot \alpha$. Second, one can produce $g \circ \gamma$. The homotopy class (relative to $\{0, 1\}$) of the first is $\phi_\alpha(f_*[\gamma])$ and the class of the second is $g_*([\gamma])$.

Writing down a homotopy between these is left as an exercise.

\[\square\]

**Corollary 6.29.** Suppose $f : X \to Y$ is a homotopy equivalence and $x_0 \in X$. Then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

**Notation 6.30.** A space $X$ is simply connected if $\pi_0(X)$ consists of a single path component and $\pi_1(X, x_0) = \{e\}$ for one, and therefore for any, basepoint $x_0 \in X$.

### 6.5 The van Kampen theorem

The basic problem is as follows. Suppose $X = U \cup V$ where $U$ and $V$ are open sets such that $U \cap V$, $U$ and $V$ are connected. Let $x_0 \in U \cap V$ be a point. Can we determine $\pi_1(X, x_0)$ from $\pi_1(U, x_0)$, $\pi_1(V, x_0)$ and $\pi_1(U \cap V, x_0)$?

The answer is yes, and in fact, we can do better. We can do the calculation for fundamental groupoids.

**Proposition 6.31.** Let $X$ be a topological space and let $U, V$ be open subspaces with $X = U \cup V$. In the diagram below, the morphisms that are not labelled are all induced by inclusions of spaces. Let $G$ be a groupoid and suppose that in the diagram below, the outer square commutes. Then
there exists a unique map of groupoids indicated by $f$ making the whole diagram commute.

\[ \xymatrix{ \Pi(U \cap V) \ar[r] \ar[d] & \Pi(U) \ar[d] \ar@/_/[dr] \ar@/^/[dl] \ar[dd]_g \\
\Pi(V) \ar[r] & \Pi(X) \ar[dl] \ar[dr]_f \ar[dd]_h \\
& \mathcal{G} } \]

**Proof.** In this case, the story about objects (the points of the spaces) is easy. The hard part is about paths.

We assume we have produced the (unique) function $f : X \to \text{ob}\mathcal{G}$. Now we have to worry about paths. Let $\gamma : I \to X$ be a path in $X$, from $a = \gamma(0)$ to $b = \gamma(1)$. We can find some decomposition of $[0,1]$ into closed subintervals $[t_0 = 0,t_1], [t_1,t_2], \ldots, [t_{r-1},1]$ such that for any $i$, the path $\gamma([t_i,t_{i+1}])$ lies either in $U$ or in $V$. Let $\gamma_i$ denote a reparametrization of $\gamma|_{[t_i,t_{i+1}]}$. This allows us to define a candidate $f([\gamma])$ as the composite of the images in $\mathcal{G}$ of $\gamma_0, \gamma_1, \ldots, \gamma_{r-1}$. We have not shown that $f([\gamma])$ is well defined, but let us observe that the definition of $f([\gamma])$ is unchanged if we replace our decomposition of $I$ by a refinement.

Observe that if a map $f : \Pi(X) \to \mathcal{G}$ of groupoids exists making the diagram commute, then it must be this one, since we have factored $[\gamma] = [\gamma_0][\gamma_1] \cdots [\gamma_{r-1}]$, and what $f$ does to $[\gamma_0], [\gamma_1], \ldots, [\gamma_{r-1}]$ is forced on us by $h$ and $g$.

Now let us prove that $f([\gamma])$ is an invariant of the homotopy class (relative to $\{0,1\}$) of $\gamma : I \to X$. That is, if we choose a possibly different path $\gamma' : I \to X$ such that $[\gamma] = [\gamma']$ and a decomposition of $\gamma'$, we obtain the same definition of $f([\gamma])$.

Suppose we have two paths $\gamma$ and $\gamma'$, a basepoint-preserving homotopy $H : I \times I \to X$ and two decompositions of $I$ as above. Using the Lebesgue covering lemma, we can find a tessellation of $I \times I$ into small rectangles $R_{ij}$ with disjoint interiors so that for each such rectangle $H|_{R_{ij}}$ lies either in $U$ or in $V$, and so that the restrictions of the tessellation to $I \times \{0\}$ and $I \times \{1\}$ refine the two decompositions of $I$. Now consider $H|_{R_{ij}}$ in each $R_{ij}$. They all give a relation in the fundamental groupoid either of $U$ or of $V$: namely $H|_{\text{bottom}} + H|_{\text{right}} = H|_{\text{left}} + H|_{\text{top}}$. Applying either $h$ or $g$, as required, each $R_{ij}$ gives us a relation in the groupoid $\mathcal{G}$. Integrating these relations together over the whole square shows that $f([\gamma]) = f([\gamma'])$.

The proof that $f$ is really a map of groupoids is not difficult. One simply has to verify that it preserves composition. The statement that it preserves composition follows by taking a composite $[\gamma][\delta]$ and decomposing each into short paths lying either in $U$ or $V$: $[\gamma_0] \cdots [\gamma_{r-1}][\delta_0] \cdots [\delta_{s-1}]$. By construction $f([\gamma][\delta])$ is the product in $\mathcal{G}$ of $f([\gamma_0]) \cdots f([\delta_{s-1}]) = f([\gamma])f([\delta])$. \qed
Corollary 6.32. Let $X$ and $U, V$ be as above. Let $A \subseteq U \cap V$ be a set of points such that each of path component of $U, V$ and $U \cap V$ contains at least one point of $A$. In the diagram below, morphisms that are not labelled are induced by inclusions of spaces. Let $\mathcal{G}$ be a groupoid, and again suppose the outer diagram commutes:

\[
\begin{array}{c}
\Pi(U \cap V, A) \longrightarrow \Pi(U, A) \\
\Pi(V, A) \longrightarrow \Pi(X, A) \\
\Pi(U, A) \longrightarrow \Pi(U) \quad \Pi(U \cap V, A) \quad \Pi(U \cap V) \quad \Pi(U \cap V, A)
\end{array}
\]

Then there exists a unique map of groupoids making the diagram commute.

Proof. For each $x \in (U \cap V) \setminus A$, choose a fixed isomorphism in $\Pi(U \cap V, A)$ from $x$ to some point $a \in A$, $\gamma : x \to a$. By means of this isomorphism, we construct a map of groupoids $\Pi(U \cap V) \to \Pi(U \cap V, A)$. We can do the same for $U$ and $V$ and $X$, and where applicable choose the same isomorphisms.

The rest of the argument follows by showing that the diagram in the corollary is a retract of the diagram in the proposition. This is a diagram chase that is best done live. We give the diagram here and advise the reader to do the chase.

Suppose the diagram with red arrows is given, then one can construct the solid cyan arrows, then using Proposition 6.31, one constructs the dashed cyan arrow. By composing, one obtains the dashed green arrow. One then verifies that this is the map $f$ asked for in the corollary. \qed
Chapter 6. The fundamental groupoid and the fundamental group

Remark 6.33. The corollary determines \( \Pi(X, A) \) up to unique isomorphism. It is the groupoid one obtains generated by the maps in \( \Pi(U, A) \) and in \( \Pi(V, A) \) subject to the relations \( \Pi(U \cap V, A) \).

Example 6.34. Cover \( S^1 \subset \mathbb{C} \) by two open sets, \( S^1 \setminus \{ \pm i \} \). Let \( A \) be the set of points \( \{ \pm 1 \} \). While \( S^1 \setminus \{ i \} \) and \( S^1 \setminus \{ -i \} \) are both contractible, \( S^1 \setminus \{ i, -i \} \) is a disjoint union of two contractible sets. The fundamental groupoids are

\[
\begin{align*}
\Pi(U, A) &= \{ f : -1 \leftrightarrow 1 : f^{-1} \} \\
\Pi(V, A) &= \{ g : -1 \leftrightarrow 1 : g^{-1} \} \\
\Pi(U \cap V, A) &= \{-1, 1\}
\end{align*}
\]

Therefore the corollary says that the fundamental groupoid of \( S^1 \) on the points \( \{ -1, 1 \} \) has two objects and is generated by two different morphisms \( f, g : -1 \to 1 \), with no relations between them other than what is forced by the groupoid axioms.

The groupoid \( \Pi(S^1, \{ 1, -1 \}) \) has two objects, 1 and \( -1 \). Let us describe the morphisms \( -1 : \to 1 \) completely, the other three sets of morphisms can be similarly described. There are two kinds of morphism \( -1 : \to 1 \):

1. strings of finite length of the form \( g f g f g f \ldots g f \).
2. strings of finite length of the form \( f^{-1} g^{-1} f^{-1} g^{-1} f^{-1} \ldots g^{-1} f^{-1} \).

In particular, if we restrict to the basepoint 1, we see that the fundamental group of \( (S^1, 1) \) is infinite cyclic generated by \( f g^{-1} \) (or by \( g^{-1} f \)).

Construction 6.35. Let \( G \) and \( H \) be two groups. A word in \( G \) and \( H \) is a string of elements \( s_1 \ldots s_n \), each one either in \( G \) or \( H \). They are subject to reduction, i.e., removing an identity element or replacing a pair \( g_1 g_2 \) by its product in \( G \), or similarly in \( H \). A reduced word is a word that cannot be reduced further.

The free product \( G \ast H \) is the group of reduced words, with concatenation-followed-by-reduction as an operation.

There are obvious homomorphisms \( G \to G \ast H \) and \( H \to G \ast H \).

Remark 6.36. Strictly speaking, the definition above presupposes that \( G \) and \( H \) are disjoint groups. If not, then we replace \( G \) and \( H \) by disjoint isomorphic groups before forming the free product.

Definition 6.37. Let \( G, H \) and \( K \) be three groups and \( \phi : K \to G, \psi : K \to H \) be homomorphisms. The notation \( G \ast_K H \) denotes the amalgamated product of \( G \) and \( H \) over \( K \). This is the quotient of \( G \ast H \) by the normal subgroup generated by elements \( \phi(k) \psi(k)^{-1} \) as \( k \) ranges over the elements in \( K \).
**Remark 6.38.** Strictly, the homomorphisms $\phi$ and $\psi$ should appear in the notation, but they do not.

As in the free case, there are homomorphisms $G \to G \ast_K H$ and $H \to G \ast_K H$. In this case, these homomorphisms make the diagram

$$
\begin{array}{c}
K \\
\downarrow \phi \\
G
\end{array} \to \begin{array}{c}
H \\
\downarrow \psi \\
G \ast_K H
\end{array}
$$

commute. If $G \ast_K H$ were replaced by $G \ast H$, then this diagram would not generally commute.

**Lemma 6.39.** Suppose there is a group $M$ and there are group homomorphisms $\alpha : G \to M$ and $\beta : H \to M$ making the diagram below commute (without the dashed arrow)

$$
\begin{array}{c}
K \\
\downarrow \phi \\
G
\end{array} \to \begin{array}{c}
H \\
\downarrow \psi \\
G \ast_K H
\end{array} \to \begin{array}{c}
M.
\end{array}
$$

Then there exists a unique homomorphism $\delta$ making the diagram commute.

**Proof.** First, consider the case where $\phi, \psi$ are trivial, so that $G \ast_K H = G \ast H$. Then each element of $G \ast H$ can be written as a word $w = g_1 h_1 \ldots g_r h_r$ (it is possible that $g_1$ or $h_r$ is the identity element). In this case, we define $\delta'(w) = \alpha(g_1) \beta(h_1) \ldots \alpha(g_r) \beta(h_r)$. It is routine to verify that this is a group homomorphism. Once that is verified, it is immediate that $\delta'$ is the unique homomorphism such that $G \to G \ast H \xrightarrow{\delta'} M$ agrees with $\alpha$ and $G \to G \ast H \xrightarrow{\delta'} M$ agrees with $\beta$.

In the general case, when $\phi$ and $\psi$ are not trivial, we construct $\delta$ as a quotient of $\delta'$. Let $N$ denote the normal subgroup of $G \ast H$ generated by words of the form $\phi(k)\psi(k)^{-1}$. Observe that $\delta'(\phi(k)\psi(k)^{-1}) = e \in M$ for all $k \in K$, so that $N \subset \ker \delta'$. Therefore, by general group theory arguments, there is an induced homomorphism: $\delta : G \ast_K H \to M$. Verifying that this makes the diagram of (6.1) commute and is unique with this property is routine and is left as an exercise. □
Corollary 6.40 (The van Kampen theorem). Let $X$ be a topological space with basepoint $x_0$, and $U, V$ path connected open subsets that cover $X$ and such that $U \cap V$ is path connected and contains $x_0$. Then $\pi_1(X, x_0) = \pi_1(U, x_0) \ast_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$, where the morphisms $\pi_1(U \cap V, x_0) \rightarrow \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \rightarrow \pi_1(V, x_0)$ are those induced by the inclusions of subspaces.

### 6.6 Examples

**Example 6.41.** Suppose $X$ is a space that can be written as the union of two open sets $U$ and $V$ where $U$ and $V$ are both simply connected and where $U \cap V$ is path connected. Let $x_0 \in U \cap V$ be a basepoint. Then by use of the van Kampen theorem:

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) \ast_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) = \{e\} \ast_{\pi_1(U \cap V, x_0)} \{e\} \cong \{e\}.$$  

This implies that $\pi_1(X, x_0) = \{e\}$.

**Example 6.42.** A special case of the previous example is the case of $\pi_1(S^n, s_0)$ whenever $n \geq 2$. To see that this case applies, consider

$$s_0 = (1, 0, 0, \ldots, 0), \quad s'_0 = (-1, 0, 0, \ldots, 0)$$

both of which are points on $S^n$. We know that $S^n \setminus \{s_0\} \approx \mathbb{R}^n$, since $S^n$ is a one-point compactification of $\mathbb{R}^n$. Similarly, $S^n \setminus \{s'_0\} \approx \mathbb{R}^n$.

For all $n$, we deduce $S^n \setminus \{s_0, s'_0\} \approx \mathbb{R}^n \setminus \{0\}$, and if $n \geq 2$, then this is path connected (it is disconnected for $n = 1$). We may then apply Example 6.41 to deduce that $\pi_1(S^n, s'_0) \cong \{e\}$ when $n \geq 2$.

We will not prove it in this course, but the spaces $S^2, S^3, \ldots$ give examples of spaces that are simply connected but not contractible.

**Example 6.43.** Suppose $(X, x_0)$ and $(Y, y_0)$ are two based spaces, and suppose $x_0 \in X$ has an open neighbourhood $U \ni x_0$ such that $\{x_0\}$ is a strong deformation retract of $U$, and similarly $y_0$ has an open neighbourhood $V$ such that $\{y_0\}$ is a strong deformation retract of $V$. A schematic picture is given in Figure 6.1.

Form the space $X \vee Y$, and denote the basepoint by $z$. Consider the two open subspaces $X \setminus V$ and $U \setminus Y$. We observe that $X \setminus X \setminus V$ is a deformation retract and $Y \setminus U \setminus Y$ is a deformation retract as well. Moreover,

$$U \setminus V = (X \setminus V) \cap (U \setminus Y)$$

is contractible (it has the basepoint as a deformation retract).

We can apply the van Kampen theorem to deduce $\pi_1(X \setminus Y, z) \cong \pi_1(X, x_0) \ast \pi_1(Y, y_0)$. 


6.6. Examples

Remark 6.44. The conditions that Example 6.43 requires of the based spaces \((X, x_0)\) and \((Y, y_0)\) are frequently satisfied in practice: for instance, if \(X\) and \(Y\) are topological manifolds, so that every point of \(X\) and \(Y\) has an open neighbourhood homeomorphic to \(\mathbb{R}^n\), then these conditions are satisfied.

We also remark that under the same hypotheses as Example 6.43, the wedge sum \(X \vee Y\) has an open neighbourhood \(U \vee V\) that admits the basepoint \(z\) as a deformation retract. Therefore, the idea in Example 6.43 can be iterated.

Example 6.45. As a special case of Example 6.43, we can calculate \(\pi_1\) of any iterated wedge sum of circles:

\[ X = \underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{k\text{-copies}}. \]

Write \(F_n\) for the free group generated by \(n\) symbols: \(x_1, x_2, \ldots, x_n\) (and no relations, so that in particular, for \(n \geq 2\), the group \(F_n\) is not abelian).

By a direct induction argument

\[ \pi_1(X, z) = \pi_1(S^1 \vee \cdots \vee S^1, z) \ast \pi_1(S^1, s_0) \cong F_{k-1} \ast F_1 \cong F_k. \]
Example 6.46. Let us show explicitly a presentation of $\pi_1(S^1 \vee S^1, z)$. In order to make the presentation more obvious, we replace $S^1 \vee S^1$ with a slightly larger space $X$, that is homotopy equivalent to it (there is an obvious deformation retract).

Two generators for $\pi_1(X, z)$ are illustrated in Figure: $\alpha$ and $\beta$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure63.png}
\caption{A space that is homotopy equivalent to $S^1 \vee S^1$, and chosen generators for $\pi_1(X, z)$.}
\end{figure}

The fundamental group in this case is not abelian, and you can verify that a loop in the class of $\alpha \beta^{-1}$ (first go around the loop on the left, then on the right, illustrated in Figure 6.4) is not homotopic (relative to endpoints) to a loop in the class of $\alpha \beta^{-1}$ (first go around the loop on the right, then on the left, illustrated in Figure 6.5).

Example 6.47. Here we consider the complement of two points, $p, q$ in $\mathbb{R}^2$. Pick any basepoint $x_0 \in \mathbb{R}^2 \setminus \{p, q\}$. The complement $\mathbb{R}^2 \setminus \{p, q\}$ is homotopy equivalent to $S^1 \vee S^1$, and so $\pi_1(\mathbb{R}^2 \setminus \{p, q\}, x_0)$ is a free group on two generators. One generator,
6.6. Examples

$\alpha$, may be taken to be a loop around $p$ in the anticlockwise direction and the other generator may be taken to be $\beta$, a loop around $q$ in the anticlockwise direction.

Figure 6.5: A loop in the class of $\beta^{-1}\alpha$.

We may consider an element in $\pi_1(\mathbb{R}^2 \setminus \{p, q\}, x_0)$ such as $w = \alpha^{-1}\beta\alpha\beta^{-1}$, but that becomes trivial if we apply any homomorphism $\pi_1(\mathbb{R}^2 \setminus \{p, q\}, x_0) \to G$ where either $\alpha \mapsto e$ or $\beta \mapsto e$. A loop in the class of $\alpha^{-1}\beta\alpha\beta^{-1}$ is illustrated in Figure 6.7. This picture (and ones like it) are often referred to as solutions to the “two nails problem”. The idea is to view the plane, $\mathbb{R}^2$, as a wall on which one wishes to hang a picture. Then two nails are hammered into the wall, at $p$ and $q$, then a piece of string is tied in a loop and wound around $p$ and $q$ as illustrated in Figure 6.7. A picture can be hung from this piece of string, and it will not fall to the floor, because the element $\alpha^{-1}\beta\alpha\beta^{-1}$ is not trivial, and therefore the string cannot be detached from the nails—if you could detach it from the nails, you would then be able to contract the loop to a point.

If one or other nail is removed, however, i.e., if either $p$ or $q$ is returned to the

Figure 6.6: $\mathbb{R}^2 \setminus \{p, q\}$ and chosen generators for $\pi_1$. 

...
Figure 6.7: A loop in the class of $\alpha^{-1} \beta \alpha \beta^{-1}$.

plane, then the loop will become contractible again. For instance, if $p$ is filled in, so the loop is viewed as representing an element in $\pi_1(\mathbb{R}^2 \setminus \{q\}, x_0) = \mathbb{Z}\beta$, by applying the homomorphism

$$\phi : \pi_1(\mathbb{R}^2 \setminus \{p, q\}, x_0) \to \pi_1(\mathbb{R}^2 \setminus \{q\}, x_0)$$

given by $\phi(\alpha) = e$ and $\phi(\beta) = \beta$. We observe that $\phi(\alpha^{-1} \beta \alpha \beta^{-1}) = \beta \beta^{-1} = e$. You can verify this yourself by visualizing a contraction of the loop to the basepoint, provided $p$ is added back to $\mathbb{R}^2$. An intermediate stage of the contraction is illustrated in Figure 6.8.

Figure 6.8: A loop representing the trivial element in $\pi_1(\mathbb{R}^2 \setminus \{q\}, x_0)$.

The effect of removing the nail at $q$ is similar, again the class of the loop becomes
6.6. Examples

Example 6.48. In this example, we calculate $\pi_1(S^1 \times S^1, x_0)$. From Proposition 5.32, we already know that $\pi_1(S^1 \times S^1, x) \cong \pi_1(S^1, s_0) \times \pi_1(S^1, s_0) \cong \mathbb{Z} \times \mathbb{Z}$, the free abelian group of rank 2. Here, we give another proof of this fact.

Cover $S^1 \times S^1$ by two open sets $U$ and $V$, as indicated in Figure 6.9. The open set $U$ is homotopy equivalent to $S^1 \lor S^1$ (it admits $S^1 \lor S^1$ as a deformation retract). The open set $V$ is contractible, since it is homeomorphic to an open ball in $\mathbb{R}^2$. The intersection $U \cap V$ is homotopy equivalent to $S^1$. This gives us the following list of homotopy groups:

- $\pi_1(U, x_0) = \langle \alpha, \beta \rangle$
- $\pi_1(V, x_0) = \{e\}$
- $\pi_1(U \cap V, x_0) = \langle \gamma \rangle$

We can now use the van Kampen theorem to present

$$\pi_1(S^1 \times S^1, x_0) \cong \pi_1(U, x_0) \ast_{\pi_1(U \cap V, x_0)} \{e\}.$$ 

In order to finish the calculation, we must determine the homomorphisms $\phi : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\psi : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$. The latter of these is trivial. We concentrate on the former. We see from Figure 6.10 that $\phi(\gamma) = \alpha \beta \alpha^{-1} \beta^{-1}$. Therefore the van Kampen theorem says that

$$\pi_1(S^1 \times S^1, x) \cong \langle \alpha, \beta | \alpha \beta \alpha^{-1} \beta^{-1} \rangle,$$

that is, the quotient of the free group by the normal subgroup generated by $\alpha \beta \alpha^{-1} \beta^{-1}$. This is a presentation of the free abelian group on two generators, which is what we expected.

Figure 6.9: Two open subsets covering the torus.
Figure 6.10: A loop $\gamma$ in $U \cap V$. 
Chapter 7

Covering spaces

7.1 Covering Spaces

Definition 7.1. A map of topological spaces \( f : Y \to X \) is a covering space map if, for all \( x \in X \), there is some open \( U \ni x \) such that the inverse image \( f^{-1}(U) \) is homeomorphic to a disjoint union \( \bigsqcup_{j \in J} V_j \) such that each induced map \( f|_{V_j} : V_j \to U \) is a homeomorphism.

Remark 7.2. Some people might require the map \( f \) to be surjective, but we do not.

Example 7.3. The prototypical examples are \( f_n : S^1 \to S^1 \) given by \( z \mapsto z^n \), and \( f_\infty : \mathbb{R} \to S^1 \) given by \( f_\infty(t) = (\cos 2\pi t, \sin 2\pi t) \).

Example 7.4. Other, more trivial, examples, include \( X \amalg X \to X \) or the inclusion of open component into a disconnected but locally connected space. These are sort of silly, so we generally concentrate in examples on the cases where both \( X \) and \( Y \) are connected.

Definition 7.5. A map of topological spaces \( f : Y \to X \) is an étale map if it is locally an open embedding: that is, if each \( y \in Y \) has an open neighbourhood \( U_y \) such that \( f|_{U_y} : U_y \to X \) is an open embedding.

Example 7.6. Both covering space maps and open embeddings are étale.

Example 7.7. Most open embeddings are not covering space maps. For instance, the inclusion of an open set \( U \hookrightarrow X \) that is not also closed is an étale map but not a covering space map (consider neighbourhoods of a point \( x \in \partial U \)).

Proposition 7.8. A covering space map is open.
Definition 7.9. Let \( f : Y \to X \) be a map and let \( x \in X \). Define the fibre of \( f \) at \( x \) to be \( f^{-1}(x) \), and denote it \( F_x \).

Notation 7.10. Let \( f : Y \to X \) be a covering space and let \( W \subseteq X \) be an open set. We say that \( f \) trivializes over \( W \) if \( f^{-1}(W) \) is a disjoint union of open sets mapping homeomorphically to \( W \).

Here comes a technical and very important proposition.

Proposition 7.11. Suppose \( f : Y \to X \) is a covering map and that \( Z \) is a space and \( i_0 : Z \to Z \times I \) is inclusion at 0 and that there are maps as indicated making the diagram (without the dashed arrow) commute:

\[
\begin{array}{ccc}
Z & \xrightarrow{G_0} & Y \\
\downarrow{i_0} & & \downarrow{f} \\
Z \times [0, 1] & \xrightarrow{g} & X
\end{array}
\]

Then there is a unique map \( G \) making both triangles commute.

A map such as \( G \) making the lower triangle commute is called a lift of \( g \).

Proof. Let \( \mathcal{W} \) be an open cover of \( X \) trivializing \( f \).

We have a cover of the compact set \( \{z\} \times I \) by the sets \( g^{-1}(W) \), where \( W \in \mathcal{W} \). Now let us choose an integer \( N > 0 \) so that for all points \((z, t) \in \{z\} \times I \), the interval \( \{z\} \times [t-1/N, t+1/N] \) is contained in some \( g^{-1}(W) \) (with the appropriate modifications at the endpoints). This is possible by the Lebesgue covering lemma.

Set \( t_i = i/N \). Each of the intervals \( \{z\} \times [t_0, t_2] \), \( \{z\} \times [t_1, t_3] \), \( \{z\} \times [t_2, t_4] \), \ldots, \( \{z\} \times [t_{N-2}, t_N] \) is contained in \( g^{-1}(W) \) for some \( W \in \mathcal{W} \).

By use of the tube lemma, for all \( i \), we can find an open \( U_i \ni z \) such that \( U_i \times [t_{i-2}, t_i] \) is contained in \( g^{-1}(W) \) for some \( W \in \mathcal{W} \). If we set \( U = \bigcap_{i=2}^{N} U_i \), then \( U \times [t_{i-2}, t_i] \) is contained in \( g^{-1}(W) \).

Now we construct the lifting map \( G \) on \( U \times I \) recursively. We already have it, in the form of \( G_0 \) on \( U \times \{0\} \). Suppose it has been extended to a lift, \( G_n \) on \( U \times [0, t_1] \).

There is a diagram

\[
\begin{array}{ccc}
U \times \{t_{i-1}\} & \xrightarrow{G_{t_{i-1}}} & Y \\
\downarrow{G_{t_i}} & & \downarrow{f} \\
U \times [t_{i-1}, t_{i+1}] & \xrightarrow{g} & X
\end{array}
\]
in which we would like to find a lift, but we know that the map \( g \) in this diagram factors through some \( W \) in \( \mathcal{W} \), to yield

\[
\begin{array}{ccc}
U \times \{ t_{i-1} \} & \xrightarrow{f} & f^{-1}(W) \\
\downarrow & & \downarrow f \\
U \times [t_{i-1}, t_{i+1}] & \xrightarrow{g} & W
\end{array}
\]

We know \( f^{-1}(W) \cong \bigsqcup_{j \in J} V_j \) where \( f|_{V_j} : V_j \to W \) is a homeomorphism. For each \( j \in J \), let \( \hat{f}_j : W \to V_j \) be the inverse of this homeomorphism.

For each \( u \in U \), there is some corresponding \( V_j \) so that \( G_{t_i}(u) \in V_j \), and since \( u \times [t_{i-1}, t_{i+1}] \) is path-connected, there is only one possible continuous lift of \( g \) on \( u \times [t_{i-1}, t_{i+1}] \), namely \( \hat{f}_j \circ g|_{u \times [t_{i-1}, t_{i+1}]} \). In fact, since \( V_j \) is open in \( f^{-1}(W) \), there is an open subset \( U_0 \ni u \) contained in \( U \) such that \( \hat{f}_j \circ g \) is the unique lift of \( g \) for all points in \( U_0 \times [t_{i-1}, t_{i+1}] \). This shows that indeed the lift we have constructed on \( U \times [t_{i-1}, t_{i+1}] \to Y \) is indeed continuous. It agrees with \( G_{t_i} \), where they are both defined, since each restricts to give a unique lift on \( U \times [t_{i-1}, t_i] \). Therefore we can use the newly-constructed lift to extend to a unique \( G_{t_{i+1}} : U \times [0, t_{i+1}] \to Y \).

Recursively, therefore, we can define a unique continuous lift in

\[
\begin{array}{ccc}
U \times \{ 0 \} & \xrightarrow{G_0} & Y \\
\downarrow & & \downarrow f \\
U \times I & \xrightarrow{g} & X
\end{array}
\]

This lift was defined on \( U \times I \) where \( U \) was an open neighbourhood of \( z \in Z \). Since \( z \) is arbitrary, we can define unique lifts on an open neighbourhood of each point in \( Z \). Since the lifts are unique, they all agree whenever they are defined, so they assemble together to produce a continuous, unique lift in

\[
\begin{array}{ccc}
Z \times \{ 0 \} & \xrightarrow{G_0} & Y \\
\downarrow & & \downarrow f \\
Z \times I & \xrightarrow{g} & X
\end{array}
\]

as required. \( \square \)

This allows us to make the following construction.
Construction 7.12. Let \( X \) be a topological space and \( A \subseteq X \). Let \( f : Y \to X \) be a covering space. For each \( a \in A \), define \( F_a = f^{-1}(a) \), which is a discrete topological space, i.e., a set. For each path \( \gamma \) in \( X \) from \( a \) to \( b \), define \( \tilde{\gamma} : F_a \to F_b \) by constructing the unique lift in

\[
\begin{array}{c}
\{0\} \xrightarrow{a} Y \\
\downarrow \gamma \\
I \xrightarrow{\Gamma} X
\end{array}
\]

and letting \( \tilde{\gamma} = \Gamma(1) \).

Proposition 7.13. The map \( \tilde{\gamma} \) defined above depends only on \([\gamma]\), the homotopy class of \( \gamma \) (rel. endpoints).

Proof. Suppose we have two paths \( \gamma \sim \delta \) in \( X \) (the homotopy being relative to endpoints). We can write down the homotopy between these by defining a function \( H : I \times I \to X \) that is \( \gamma \) on \( I \times \{0\} \) and is a rescaling of \( \delta \) on the other three edges. Then uniqueness of lifting ensures that the lift \( \Gamma \) is homotopic (relative to endpoints) to \( \Delta \), the lift of \( \delta \). □

In order to keep the notation from becoming littered with \([\cdot]\) symbols, we will write \( \gamma \) instead of \([\gamma]\) from now on. We will have to rely on context to determine when \( \gamma \) (the path) is meant and when \([\gamma]\) (the homotopy class of the path) is meant, but this will turn out not to be difficult to do.

Now come two small, but very helpful, results about lifting compositions of paths.

Proposition 7.14. Suppose \( f : Y \to X \) is a covering space and \( \gamma, \delta \) are two paths in \( X \) such that \( \gamma(1) = \delta(0) \). Then the two functions

\[
\delta \circ \tilde{\gamma} : F_{\gamma(0)} \to F_{\delta(1)}
\]

and

\[
(\gamma \cdot \delta) : F_{\gamma(0)} \to F_{\delta(1)}
\]

agree.

Proof. Choose a point \( y \in F_{\gamma(0)} \). Let \( \Gamma : I \to Y \) denote the unique lift of \( \gamma \) to \( Y \) that starts at \( y \). Let \( \Delta : I \to Y \) be the unique lift of \( \delta \) to \( Y \) that starts at \( \Gamma(1) \). Then \( \Gamma \cdot \Delta \) is defined and is in fact a lift of \( \gamma \cdot \delta \) that starts at \( y \). It is therefore the unique such lift. We now can compare:

\[
\delta \circ \tilde{\gamma}(y) = \Delta(1) = \Delta \cdot \Gamma(1) = \tilde{\gamma} \cdot \delta(y),
\]

which is what we wanted. □
Chapter 7. Covering spaces

Proposition 7.15. Suppose $f : Y \to X$ is a covering space, and $e_x : I \to X$ is the constant path at $x \in X$. Then $\tilde{e}_x : F_x \to F_x$ is the identity map.

Proof. If $y \in F_x$, then the unique lift of $e_x$ starting at $y$ is $e_y$. □

Corollary 7.16. Suppose $f : Y \to X$ is a covering space and $\gamma : I \to X$ is a path, then $\tilde{\gamma} : F_{\gamma(0)} \to F_{\gamma(1)}$ is a bijection.

In particular, if $X$ is path-connected, then every fibre is in bijective correspondence with every other fibre.

Proof. There is an inverse to $\tilde{\gamma}$ given by $\tilde{\gamma}^{-1}$, since $\gamma \cdot \gamma^{-1}$ and $\gamma^{-1} \cdot \gamma$ are both constant paths, and we can now use Propositions 7.14 and 7.15. □

7.2 Fundamental group action

In the previous section, we saw that if $f : Y \to X$ is a covering space, then a path $\gamma$ in $X$ induces a bijection $F_{\gamma(0)} \to F_{\gamma(1)}$. In particular, when $\gamma$ is a loop based at $x \in X$, we obtain an automorphism of $F_x$. These automorphisms are compatible with composition of paths, according to Proposition 7.14, although there is a reversal of the order.

Construction 7.17. Suppose $f : Y \to X$ is a covering space and $x \in X$ is a point. For each $\gamma \in \pi_1(X, x)$ and each $y \in F_x$, define $y \cdot \gamma$ to be $\tilde{\gamma}(y) \in F_x$, i.e., the endpoint of the path in $Y$ obtained by lifting $\gamma$, starting at $y$.

The assignment $(y, \gamma) \mapsto y \cdot \gamma$ is a function $F_x \times \pi_1(X, x) \to F_x$.

Proposition 7.18. The function $F_x \times \pi_1(X, x) \to F_x$ defined above is a right action of $\pi_1(X, x)$ on $F_x$.

Proof. To prove this is a right action, we have to prove two things:

First, we want to show $y \cdot e_x = y$, where $e_x$ denotes the constant loop at $x$. This is Proposition 7.15.

Second, $(y \cdot \gamma) \cdot \delta = y \cdot (\gamma \delta)$. This is a special case of Proposition 7.14. □

Notation 7.19. This action of $\pi_1(X, x)$ on the fibre over $x$ is called the monodromy action.

Remark 7.20. There is a category of covering spaces of $X$ where the objects are maps $f : Y \to X$ and the maps are maps $h : Y \to Y'$.
making the diagram commute. Write this category as $\text{Cov}_X$.

There is an obvious category of right $\pi_1(X, x_0)$-sets, written $\text{Set-}\pi_1(X, x_0)$.

We now discuss the relationship, for a covering space $f : X \to Y$, between $\pi_1(Y, y)$, $\pi_1(X, x)$ and $F_x$.

**Proposition 7.21.** Let $f : Y \to X$ be a covering space and suppose $y \in Y$. Define $x = f(y)$. The homomorphism $f_* : \pi_1(Y, y) \to \pi_1(X, x)$ is injective.

**Proof.** The proof is all about uniqueness of lifting.

Write $e_x$ for the constant path at $x$.

It will suffice to show that the kernel of $f_*$ is trivial. Suppose $\gamma$ is a loop in $Y$, based at $y$, such that $f_*(\gamma) = e_x$ in $\pi_1(X, x)$. We want to show that $\gamma$ represents the same element as $e_y$ in $\pi_1(Y, y)$.

Then $f \circ \gamma : I \to X$ is homotopic, relative to $\{0, 1\}$, to the constant path at $x$. Let $H : I \times I \to X$ be a specific homotopy from $f \circ \gamma$ to $e_x$. Then lift, as in the diagram

\[
\begin{array}{ccc}
I & \xrightarrow{\gamma} & Y \\
\downarrow_{i_0} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
\end{array}
\]

Now let us consider what we know about $\tilde{H} : I \times I \to Y$:

1. $\tilde{H}(t, 0) \to Y$ is $\gamma$, since the upper triangle commutes.

2. $\tilde{H}(0, \cdot)$ and $\tilde{H}(1, \cdot)$ are lifts of $H(0, \cdot)$ and $H(1, \cdot)$, which are both $e_x$. Since $e_y$ is a possible lift of $e_x$, uniqueness of lifts imply that $\tilde{H}(0, s) = e_y(s) = \hat{H}(1, s)$.

3. $\tilde{H}(\cdot, 1)$ is a lift of $H(\cdot, 1)$, which is $e_x$. Again, uniqueness of lifts says that $\tilde{H}(t, 1) = y$ for all $t$.

In summary, $\tilde{H}$ is a homotopy (relative to $\{0, 1\}$) from $\gamma$ to $e_y$. Therefore $\gamma$ represents the identity element of $\pi_1(Y, y)$, which is what we wanted to show. \qed

**Construction 7.22.** Suppose $f : Y \to X$ is a covering space. Choose a basepoint $y \in Y$ and set $x = f(y)$. Define a function $\partial : \pi_1(X, x) \to F_x$ by $\partial(\gamma) = yy\gamma$.

Give $\pi_1(X, x)$ the right $\pi_1(X, x)$-action defined by $\gamma \cdot \gamma' = \gamma\gamma'$. It is immediate that $\partial$ is a function of sets with right $\pi_1(X, x)$-action, since $\partial(\gamma\gamma') = yy\gamma'$.

**Proposition 7.23.** Using the notation of Construction 7.22,
1. The image of \( \partial \) consists of the elements \( y' \in F_x \) such that \( y' \) is in the same path component as \( y \).

2. The inverse image \( \partial^{-1}(y) \) is \( \text{im } f_* \) from Proposition 7.21.

Proof.  
1. Let \( y, y' \in F_x \) be points. Suppose there is a path \( \gamma : y \rightarrow y' \) in \( Y \). Consider the loop \( f(\gamma) \). By construction, \( \partial(f(\gamma)) = y' \).

Conversely, suppose \( \partial(\gamma) = y' \), then there is a lift of \( \gamma \) to a path in \( Y \) starting at \( y \) and ending at \( y' \).

2. Suppose \( \partial(\gamma) = y \), then the unique lift \( \Gamma \) of \( \gamma \) to a path starting at \( y \) is actually a loop that starts and ends at \( y \). In particular \( f_*(\Gamma) = \gamma \). This shows that \( \partial^{-1}(y) \subset \text{im}(f_*) \).

To show the reverse inclusion, suppose \( \gamma = f_*(\Gamma) \). Then \( \Gamma \) is a loop starting at \( y \), and is the unique lift of \( \gamma \) to a path starting at \( y \). Therefore \( \partial(\gamma) = \Gamma(1) = y \).

\( \square \)

Remark 7.24. This proposition is stated using the function \( \partial : \pi_1(X,x) \rightarrow F_x \). Another way to state it is to use the language of group actions. It is this:

1. the orbit of \( y \in F_x \) under the \( \pi_1(X,x) \)-action consists of those points \( y' \in F_x \) that lie in the same path component of \( Y \) as \( y \).

2. the stabilizer of \( y \in F_x \) under the \( \pi_1(X,x) \)-action is \( \text{im}(f_*) \).

Remark 7.25. Even though \( F_x \) does not (in general) have any algebraic structure beyond being a right \( \pi_1(X,x) \)-set, you can still view \( y \in F_x \) as the “trivial element”. The preceding result says that the “kernel” of \( \partial \) is exactly the image of \( f_* \).

7.3 Fundamental groups of quotients

Definition 7.26. A right action of a (discrete) group \( G \) on a space \( Y \) is a covering space action if, for all \( y \in Y \) there exists an open set \( U \ni y \) such that \( U \cap Ug \neq \emptyset \) implies \( g = e \).

Remark 7.27. Although we have seen a number of right actions in this course so far, you should be aware that left actions are more common. The difference between a left- and a right-\( G \) action on a space \( X \) is how \( g_1g_2 \) acts on an element \( x \). In a right action, this is the same as acting by \( g_1 \) and then by \( g_2 \), i.e., \((xg_1)g_2\), whereas in a left action, we act by \( g_2 \) first and then by \( g_1 \), i.e., \( g_1(g_2x) \).

There is a standard way to convert a right-\( G \)-action into a left-\( G \)-action: define \( g \cdot x = xg^{-1} \). A similar idea will convert a left-\( G \)-action into a right-\( G \)-action.
Remark 7.28. Suppose $Y$ is a Hausdorff space and that $G$ is a finite group acting on $Y$ on the right with the property that $yg \neq y$ unless $g = e$, i.e., that the action is free. Then for each $y \in Y$ and each $g \in G \setminus \{e\}$, we can find $U_g \ni y$ and $V_g \ni yg$ open such that $U_g \cap V_g = \emptyset$. Let $W_g = U_g \cap V_g g^{-1}$. Then form $W = \bigcap_{g \in G \setminus \{e\}} W_g$ which is open. Suppose $z \in W \cap W_g$ for some $g$, then $z \in W_g$, so that $z \in U_g$ and $zg \in V_g$, a contradiction.

We conclude that a free action by a finite group $G$ is a covering space action.

Proposition 7.29. If $Y$ is a topological space and $G$ is a group acting on $Y$ (on the right) by means of a covering space action, then the quotient map $q : Y \to Y/G$ is a covering space map.

Proof. For all $x \in Y/G$, we can choose a preimage $y \in q^{-1}(Y)$ and an open neighbourhood $U \ni y$ meeting the condition that $U \cap U_g = \emptyset$ unless $g = e$. Then $q(U) = W$ is an open neighbourhood of $x$ and $q^{-1}(W) \approx \bigsqcup_{g \in G} U_g$. Furthermore, for all $g \in G$, the map $q|_{U_g} : U_g \to W$ is an open bijection, i.e., a homeomorphism, as required. \qed

Definition 7.30. A covering space $q : Y \to Y/G$ that arises from a covering space action is called a normal covering space.

Example 7.31. Our first example of a covering space, $f : \mathbb{R} \to S^1$, is a normal covering space. It comes from the action of the group $\mathbb{Z}$ on $\mathbb{R}$ given by $(n, x) \mapsto n + x$.

Example 7.32. Not all covering spaces are normal covering spaces. An example where $X$ is not a quotient of $Y$ by a group action is given in Figure 7.1. In the figure, a (topological) graph is depicted. The edges labelled $\alpha$ map to the edge labelled $\alpha$ and similarly for $\beta$. The vertices map to the unique vertex in the target. The target $X$ is not the quotient of $Y$ by a group action. We will return to this topic later.

Figure 7.1: A covering space that is not a normal covering.
Chapter 7. Covering spaces

Construction 7.33. Suppose \( q : Y \to Y/G \) is a quotient by a covering space action. Write \( X = Y/G \). Choose a basepoint \( y \in Y \), and write \( x = q(y) \). The fibre, \( F_x \), is precisely the set of points \( \{yg\}_{g \in G} \). There is an obvious bijection \( F_x \to G \) given by \( yg \mapsto g \).

Recall the function \( \partial : \pi_1(X,x) \to F_x \) defined in Construction 7.22. Compose this with the bijection \( F_x \to G \) just defined to yield a function: \( \partial' : \pi_1(X,x) \to G \).

Proposition 7.34. Suppose \( f : Y \to Y/G \) is a normal covering space. Write \( X = Y/G \). Let \( y \in Y \) be a point and set \( x = f(y) \). The function \( \partial' : \pi_1(X,x) \to G \) is a homomorphism of groups.

Proof. Suppose \( \gamma \) and \( \gamma' \) are two elements of \( \pi_1(X,x) \). We must show that \( \partial' (\gamma' \cdot \gamma) = \partial' (\gamma') \partial' (\gamma) \).

As is our custom, use \( \gamma \) and \( \gamma' \) also to denote loops \( \gamma : I \to X \) and \( \gamma' : I \to X \) representing these elements. The definition of \( \partial' (\gamma) \in G \) is as follows: Let \( \Gamma \) be a lift of \( \gamma \) to \( Y \) starting at \( y \). It ends at \( \Gamma(1) = yg \) for some \( g \), and this \( g \) is \( \partial' (\gamma) \). Similarly, to define \( \partial' (\gamma') \), construct \( \Gamma' \) and set \( \Gamma'(1) = y\partial' (\gamma') \), and in order to define \( \partial' (\gamma' \cdot \gamma) \), construct the lift \( \Delta \) of \( \gamma' \cdot \gamma \) starting at \( y \) then \( \Delta(1) = y\partial' (\gamma' \cdot \gamma) \).

Since \( \partial' (\gamma) \) is an element of \( G \), and \( G \) acts on \( Y \), there is a homeomorphism \( Y \to Y \) given by \( y \mapsto \partial' (\gamma) \). Composing this with \( \Gamma' : I \to Y \), we obtain a path \( \Gamma' \partial' (\gamma) : I \to Y \) that starts at \( y\partial' (\gamma) \) and ends at \( y\partial' (\gamma') \partial' (\gamma) \). Note now that \( (\Gamma' \partial' (\gamma)) \cdot \Gamma \) is a well-defined path in \( Y \) starting at \( y \) and lifting \( \gamma' : \gamma \). Since lifts of paths are uniquely determined by their starting point, it must be the case that \( \Delta = (\Gamma' \partial' (\gamma)) \cdot \Gamma \).

Consider now the endpoint of this lift:

\[
y \partial' (\gamma' \cdot \gamma) = \Delta(1) = ((\Gamma' \partial' (\gamma)) \cdot \Gamma)(1) = y\partial' (\gamma') \partial' (\gamma).
\]

This implies that \( bd' (\gamma' \cdot \gamma) = \partial' (\gamma') \partial' (\gamma) \), which is what we wanted to show. \( \square \)

Corollary 7.35. If \( f : Y \to Y/G = X \) is a normal covering space and \( y \in Y \) is a chosen basepoint, then the image of the injective homomorphism \( f_* : \pi_1(Y,y) \to \pi_1(X,x) \) is precisely the kernel of \( \partial' : \pi_1(X,x) \to G \). In particular \( \text{im}(f_*) \) is a normal subgroup of \( \pi_1(X,x) \).

We may see later that under mild hypotheses on \( X \), every path-connected covering space \( f : Y \to X \) for which \( \text{im}(f_*) \) is a normal subgroup of \( \pi_1(X,x) \) comes about as a quotient \( Y \to Y/G = X \).

Example 7.36. Whenever we have a simply-connected space \( Y \) and a covering space action of a group \( G \) on \( Y \), then for any point \( x \in X = Y/G \), there is an isomorphism \( \pi_1(X,x) \cong G \). This is because the homomorphism \( \partial' \) is surjective (because \( Y \) is path connected) and injective, because the kernel is the image of \( \pi_1(Y,y) \), which is trivial.
**Definition 7.37.** A covering space \( f : Y \to X \) where \( Y \) is simply connected is called a universal covering space.

**Example 7.38.** The space \( \mathbb{R} \) is simply connected and the group \( \mathbb{Z} \) acts on it by \((x, n) \mapsto x + n\). The quotient map is \( f : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \approx S^1 \). This gives another proof that \( \pi_1(S^1, *) \cong \mathbb{Z} \).

**Example 7.39.** Consider the (topological) group \( C^\mathbb{C} = \mathbb{C} \setminus \{0\} \). Let \( n \) be a nonzero integer, and define a map \( f_n : C^\mathbb{C} \to C^\mathbb{C} \) by \( z \mapsto z^n \). The fibre over 1 of this map is the finite, discrete group of \([n]\)-th roots of 1. This group is isomorphic to \( \mathbb{Z}/(n) \).

There is an action of \( \mathbb{Z}/(n) \) on \( C^\mathbb{C} \) given by \((z, \xi) = z\xi\). The map \( f_n : C^\mathbb{C} \to C^\mathbb{C} \) factors through the quotient \( q : C^\mathbb{C} \to C^\mathbb{C}/\mathbb{Z} \) (use the universal property of quotients), and the induced map \( h : C^\mathbb{C}/(\mathbb{Z}/(n)) \to C^\mathbb{C} \) is a continuous bijection. It is also open, which follows because \( q \) is open and \( f_n \) is continuous.

In particular \( f_n : C^\mathbb{C} \to C^\mathbb{C} \) is a normal covering space. Give both source and target the basepoints 1. The group \( \pi_1(C^\mathbb{C}, 1) \) is infinite cyclic, since \( C^\mathbb{C} \approx \mathbb{Z} \). Fix a generator \( \alpha \in \pi_1(C^\mathbb{C}, 1) \) represented by the loop \( I \to C^\mathbb{C} \) given by \( t \mapsto (\cos(2\pi t), \sin(2\pi t)) \). There is an injection \( f_* : \pi_1(C^\mathbb{C}, 1) \to \pi_1(C^\mathbb{C}, 1) \) and the quotient is isomorphic to \( \mathbb{Z}/(n) \). We deduce that \((f_n)_*(\alpha) = n\alpha \) (some attention is required to distinguish between \( na \) and \(-na\)).

Now suppose \( n \geq 2 \). We will show that there is no continuous \( n\)-th root function defined on all of \( C^\mathbb{C} \). Suppose there were such a function \( s : C^\mathbb{C} \to C^\mathbb{C} \). Replace \( s \) by \( s'(z) = s(z)/s(1) \). This function \( s' \) is also a continuous \( n\)-th root function and it has the property that \( s'(1) = 1 \). By hypothesis on \( S \), it is the case that \( f \circ s' = \text{id}_{C^\mathbb{C}} \). This implies that \( f_* \circ s_* = \text{id} : \pi_1(C^\mathbb{C}, 1) \to \pi_1(C^\mathbb{C}, 1) \), but \( f_* \) is not surjective, a contradiction.

Here is a small note about when a quotient of a Hausdorff space is Hausdorff.

**Definition 7.40.** Let \( Y \) be a locally compact Hausdorff space and suppose \( \alpha : Y \times G \to Y \) is a right \( G \) action. We say that the action of \( G \) on \( Y \) is proper if the following holds:

- for all compact subsets \( K \) of \( Y \), the set of \( g \in G \) such that \( K \cap Kg \neq \emptyset \) is finite.

**Remark 7.41.** A map \( f : A \to B \) of locally compact Hausdorff spaces is said to be proper if \( f^{-1}(K) \) is compact whenever \( K \) is a compact subset of \( B \). Associated to the action map of spaces above, there is a map \( \Phi : Y \times G \to Y \times Y \) given by \( \Phi(y, g) = (y, yg) \). It is an exercise to verify that the action is proper in the sense above if and only if \( \Phi \) is proper in the sense defined here.

**Remark 7.42.** An action by a finite group is always proper.

**Proposition 7.43.** Suppose \( \alpha : Y \times G \to Y \) is a free proper action by a group \( G \) on a locally compact Hausdorff space \( Y \). Then \( \alpha \) is a covering space action and the quotient space \( X = Y/G \) is locally compact and Hausdorff.
Write \( q : Y \to X \) for the quotient map.

**Proof.** First we show the action is a covering space action. Let \( y \in Y \) be a point and choose an open \( U \ni y \) with compact closure. By virtue of the proper hypothesis on the action, there are only finitely many \( g \in G \) such that \( U \) and \( Ug \) intersect. Denote the set of such \( g \) by \( A \). For each \( g \in A \), it is possible to find disjoint open neighbourhoods \( W_g \ni y, V_g \ni yg \). Then form \( U_g = U \cap W_g \cap V_g g^{-1} \), which is open. Note that \( U_g \cap U_{g'} = \emptyset \).

Form \( U' = \cap_{g \in A} U_g \). We leave it as an exercise to establish that \( U' \cap U'g = \emptyset \) for all \( g \in G \).

Being locally compact is a local condition on a space: i.e., \( X \) is locally compact if and only if every point has a locally compact neighbourhood. Therefore, if \( q : Y \to X \) is a surjective covering space, then \( Y \) is locally compact if and only if \( X \) is locally compact.

It remains to show that the quotient is Hausdorff. Suppose \( x_1 \) and \( x_2 \) are two distinct points in \( X \) and let \( y_1 \) and \( y_2 \) be points in \( Y \) such that \( q(y_i) = x_i \). We can choose an open \( U_1 \) containing \( y_1 \) and \( U_2 \) containing \( y_2 \) so that each has compact closure. In particular, only finitely many of the translates \( U_2g \) meet \( U_1 \). As before, we can replace \( U_1 \) and \( U_2 \) by smaller open neighbourhoods of \( y_1 \) and \( y_2 \), now having the property that \( U_1 \) and \( U_2g \) are disjoint for all \( g \in G \). We observe that this implies that \( U_1g \cap U_2g' = \emptyset \) for all \( g, g' \in G \), since \( x \in U_1g \cap U_2g' \) implies \( xg^{-1} \in U_1 \cap U_2g'g^{-1} \).

Let \( V_1 = \bigcup_{g \in G} U_1g \) and \( V_2 = \bigcup_{g \in G} U_2g \). These are two open sets, and they are saturated in that \( q^{-1}(V_1) = V_1 \) and similarly for \( V_2 \). In particular, \( q(V_1) \) is an open neighbourhood of \( x_1 \) and \( q(V_2) \) is an open neighbourhood of \( x_2 \) and their intersection is empty, as required. \( \square \)

### 7.4 Lifting and covering spaces

**Proposition 7.44.** Let \( f : Y \to X \) be a covering space and let \( y_0 \in Y \) be a basepoint and \( x_0 = f(y_0) \). Let \((Z, z_0)\) be a pointed space where \( Z \) is path connected and locally path connected. Let \( g : (Z, z_0) \to (X, x_0) \) be a map. Suppose \( \text{im}(g_* : \pi_1(Z, z_0) \to \pi_1(X, x_0)) \subseteq \text{im}(f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)) \). Then there exists a unique lift \( h \) making the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & \nabla \\
\downarrow{f} & & \downarrow{f} \\
Z & \xrightarrow{g} & X
\end{array}
\]

commute.
Proof. Define $h$ as a function as follows: if $z \in Z$, then let $y$ be a path in $Z$ from $z_0$ to $z$. The image under $g$ of $y$ is a path in $X$ from $x_0$ to $g(z)$. This path has a unique lift to a path $\tilde{y}$ from $y_0$ to $\tilde{y}(1)$. Define $h(z) = \tilde{y}(1)$.

This definition seems to depend on the choice of $y$. Let us show that it is really independent of that choice. Suppose $\delta$ is a path in $Z$ from $z_0$ to $z$, possibly different from $y$. Then $\delta \gamma_{\leftarrow}$ is a loop in $Z$, and so $g(\delta \gamma_{\leftarrow})$ is a loop in $X$. The hypothesis on the fundamental groups implies that $g(\delta \gamma_{\leftarrow}) \in \pi_1(X, x_0)$ acts trivially on $y_0$, so that $(\delta \gamma_{\leftarrow})$ lifts to a loop, based at $y_0$. By uniqueness of lifting, this loop must be $\tilde{\delta} \tilde{\gamma}_{\leftarrow}$, so that in particular, $\tilde{\delta}(1) = \tilde{\gamma}(1)$. This implies that the definition of $h(z)$ was independent of the choice of path.

Now we must show that the function $h$ that we defined is continuous. Suppose $W \subseteq Y$ is an open set and $h(z) \in W$. We can find an open neighbourhood $U \ni h(z)$ and $U \subseteq W$ such that $f|_U$ is a homeomorphism onto an open set. Note that $g(z) \in f(U)$. Since $g$ is continuous, there is an open set $V \ni z$ such that $V$ is path connected and such that $g(V) \subseteq f(U)$. Take any $z' \in V$. There exists a path $\varepsilon$ from $z$ to $z'$ in $V$, and by composing $g \varepsilon$, we obtain a path from $z_0$ to $z'$. We may use this path to calculate $h(z') = \tilde{\gamma} \varepsilon(1)$. By uniqueness of lifting, $\varepsilon$ lies entirely within $U$, and therefore $h(z') = \tilde{\gamma} \varepsilon(1)$ lies in $U$. It follows that $z \in U \subseteq h^{-1}(W)$. Since $z$ was arbitrary in $W$, the function $h$ is continuous.

Finally, we show that $h$ is unique. Suppose $h'$ is another lift and $h(z) \neq h'(z)$. Let $\gamma : I \to Z$ be a path from $z_0$ to $z'$, then $h \circ \gamma$ and $h' \circ \gamma$ give two different lifts of the path $g \circ \gamma$ both starting at $y_0$. This is impossible. \hfill $\square$

### 7.5 The classification of covering spaces

We have seen that if $f : Y \to X$ is a covering space and $x \in X$, then the fibre $f^{-1}(x)$ is a (discrete) set with a right $\pi_1(X, x)$-action.

**Definition 7.45.** Let $X$ be a topological space and $f_1 : Y_1 \to X$, $f_2 : Y_2 \to X$ be two covering spaces. A map of coverings spaces $h : Y_1 \to Y_2$ is a map of spaces that also satisfies $f_1 = f_2 \circ h$, i.e., it makes

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{h} & Y_2 \\
\downarrow_{f_1} & & \downarrow_{f_2} \\
X & & 
\end{array}
\]

commute.

**Proposition 7.46.** If $h : Y_1 \to Y_2$ is a map of covering spaces and $x \in X$ is a point, then there is an induced function $h_x : f_1^{-1}(x) \to f_2^{-1}(x)$ given by $h_x(y) = h(y)$. This function is
compatible with the right $\pi_1(X, x)$-actions on the fibres in the sense that

$$h(y \cdot \gamma) = h(y) \cdot \gamma$$

**Proof.** To form $y \cdot \gamma$, we lift $\gamma$ to a path $\Gamma$ in $Y_1$ starting at $y$. Then $h \circ \Gamma$ is the unique lift of $\gamma$ in $Y_2$ starting at $h(y)$.

$$h(y) \cdot \gamma = h \circ \Gamma(1) = h(y \cdot \gamma).$$

□

**Remark 7.47.** That is, a map of covering spaces yields an induced morphism of fibres as sets with right $\pi_1(X, x)$-action. It is easy to verify that

1. The morphism on fibres induced by the identity is the identity.

2. If $h_1$ and $h_2$ are two composable maps of covering spaces, then $(h_2)_x \circ (h_1)_x = (h_2 \circ h_1)_x$.

Taken together, these imply that we have a functor

Fibre at $x$: Covering spaces of $X \to$ Sets with right $\pi_1(X, x)$-action.

Under mild hypotheses on $X$ (including path connectedness), this functor tells you everything about the covering spaces of $X$:

1. All sets with a right $\pi_1(X, x)$-action arise as the fibre of some covering space (Proposition 7.52).

2. All morphisms of sets with right $\pi_1(X, x)$ arise from maps of covering spaces (Proposition 7.50).

3. If $h_1, h_2 : Y_1 \to Y_2$ are two maps of covering spaces that give rise to the same map of fibres, then $h_1 = h_2$ (Proposition 7.49).

In particular this implies that if $h : Y_1 \to Y_2$ is a map of covering spaces such that the induced map on the fibres over $x$ is a bijection, then $h$ is a homeomorphism. That is, the fibre over $x$ determines the covering space up to homeomorphism.

**Remark 7.48.** The term for the situation above is that there is an equivalence of categories between the category of covering spaces of $X$ and
Proposition 7.49. Suppose $X$ is a path connected space with basepoint $x$. Suppose $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are two covering spaces and write $F_1$ and $F_2$ for the fibres of each over $x$. Suppose $h_1, h_2 : Y_1 \to Y_2$ are two maps of covering spaces of $X$ such that the two induced functions on fibres $(h_i)_x : F_1 \to F_2$ agree. Then $h_1 = h_2$.

Proof. Let $y \in Y_1$ be a point (not necessarily in the fibre), and let $\gamma$ be a path in $X$ from $x$ to $f_1(y)$. We can lift $\gamma$ uniquely to a path $\Gamma$ in $Y_1$ ending at $y$, and taking the starting point of this path gives us $y_0 \in Y_1$, a basepoint for $Y$ in the fibre $F_1$. By hypothesis, $h_1(y_0) = h_2(y_0)$, and now $h_1 \circ \Gamma$ and $h_2 \circ \Gamma$ are two lifts of $\gamma$ to $Y_2$, both starting at the same point, $h_1(y_0)$. Uniqueness of lifts now implies that $h_1 \circ \Gamma = h_2 \circ \Gamma$, so that in particular their endpoints agree, i.e. $h_1(y) = h_2(y)$. Since $y \in Y$ was an arbitrary point, we have proved the proposition. \hfill $\Box$

Proposition 7.50. Suppose $X$ is a path connected and locally path connected space with basepoint $x$. Suppose $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are two covering spaces and write $F_1$ and $F_2$ for the fibres of each over $x$. Suppose $\phi : F_1 \to F_2$ is a function between the fibres that is compatible with the right $\pi_1(X, x)$-actions. Then there is a map $h : Y_1 \to Y_2$ of covering spaces inducing $\phi$.

Proof. The bulk of this has been done already. Note that the previous proposition states that $h$ is unique if it exists.

Choose a point $y \in F_1$. We define $h$ on the path component of $y$ in $Y_1$. There is no loss of generality in supposing $Y_1$ actually path-connected, since the $\pi_1(X, x)$-action restricts to an action on the intersection of $F_1$ with the path component of $y$. Similarly, we may replace $Y_2$ by the path component of $\phi(y)$.

In particular, once we assume $Y_1$ and $Y_2$ are path connected, we know that $F_1$ and $F_2$ each consist of a single orbit for the $\pi_1(X, x)$-action.

Use $y$ as a basepoint for $Y_1$ and $\phi(y)$ as a basepoint for $Y_2$. Then we can define an injective homomorphism $(f_1)_* : \pi_1(Y_1, y) \to \pi_1(X, x)$ and similarly for $(f_2)_*$.

There is an isomorphism of sets with a right $\pi_1(X, x)$-action $F_1 \cong \pi_1(X, x)/(\mathrm{im}(f_1)_*)$ and similarly there is an isomorphism of sets with a right $\pi_1(X, x)$-action $F_2 \cong \pi_1(X, x)/(\mathrm{im}(f_2)_*)$. The existence of a morphism $\phi$ of sets with $\pi_1(X, x)$-action implies that any element of $\pi_1(X, x)$ that acts trivially on $y$ must also act trivially on $\phi(y)$. Therefore, $\mathrm{im}(f_2)_* \subset \mathrm{im}(f_1)_*$. We may then use Proposition 7.44 to produce a (unique) map $h$ making the diagram

\[
\begin{array}{ccc}
2 & \to & 1 \\
\downarrow & & \downarrow \\
X & & X
\end{array}
\]

commute and satisfying $h(y) = \phi(y)$. 

7.5. The classification of covering spaces
It remains to show that \( h \) really does yield \( \phi \) when restricted to the fibres \( F_1 \) and \( F_2 \). To show this, suppose \( \gamma \in \pi_1(X, x) \). Since \( F_1 \) consists of a single orbit, it will suffice to show that \( h(y \cdot \gamma) = h(y) \cdot \gamma \). First consider \( y \cdot \gamma \). This is the element \( \Gamma(1) \), where \( \Gamma \) is a lift of \( \gamma \) to \( Y_1 \) starting at \( y \). Then consider \( h \cdot \Gamma \), which is a lift of \( \gamma \) to a path starting at \( \phi(y) \). Therefore \( h \cdot \Gamma(1) = \phi(y) \cdot \gamma = h(y) \cdot \gamma \), which is what we wanted. \( \square \)

**Corollary 7.51.** Let \( X \) be a path connected and locally path connected space with basepoint \( x \). Suppose \( f_1 : Y_1 \to X \) and \( f_2 : Y_2 \to X \) are covering spaces such that the fibres \( F_1 \) and \( F_2 \) are isomorphic as sets with right \( \pi_1(X, x) \)-action. Then \( Y_1 \) and \( Y_2 \) are isomorphic as covering spaces of \( X \).

**Proof.** There exists an abstract isomorphism \( \phi : F_1 \to F_2 \) and an inverse \( \phi^{-1} : F_2 \to F_1 \). Using Proposition 7.49, form the maps of covering spaces associated to each of these: \( \phi \) is associated to \( h \) and \( \phi^{-1} \) is associated to a map \( h' \) that satisfies \( h \circ h' = \text{id}_{F_2} \) and \( h' \circ h = \text{id}_{F_1} \) by virtue of Proposition 7.49. This implies that \( h' = h^{-1} \), so \( h \) is an isomorphism of covering spaces, as required. \( \square \)

**Proposition 7.52.** Suppose \( X \) is a path connected and locally simply connected space and let \( x \in X \) be a basepoint. Let \( S \) be a set with a right \( \pi_1(X, x) \)-action. There is a covering space \( f : Y \to X \) where the fibre \( f^{-1}(x) \) is isomorphic to \( S \) with the given right \( \pi_1(X, x) \)-action.

**Proof.** If \( S \) is empty, there is nothing to do, so assume \( S \) is not empty.

As before, it suffices to consider the case where \( S \) consists of a single orbit, since a covering space \( f : Y \to X \) is made up of disjoint path components, each corresponding to a different orbit in the fibre.

Let \( y \in F \) be a basepoint. Consider \( K \), the stabilizer of the \( \pi_1(X, x) \) action on \( y \).

Now impose an equivalence relation on paths \( \gamma : I \to X \) such that \( \gamma(0) = x \); we say \( \gamma \sim \gamma' \) if \( [\gamma'] \circ [\gamma^{-1}] \in K \). In particular \( \gamma \sim \gamma' \) implies that \( \gamma(1) = \gamma'(1) \). Let \( Y \) denote the set of equivalence classes of such paths and denote the class of \( \gamma \) by \([\gamma]\).[8]

We generate a topology on \( Y \) as follows. Choose a point \([\gamma]\). Write \( x' = \gamma(1) \). Find \( W \ni x' \) that is simply connected—any \( x' \in X \) has a local base consisting of such sets. For each \( z \in W \), there exists a path \( \delta \) from \( x' \) to \( z \). The class of \([\gamma \delta]\) is independent of the choice of \( \delta \) by virtue of the local simple-connectivity of \( X \); any alternative choice of \( \delta' \) would give us \( \delta' \delta'^{-1} \approx e_{x'} \). Define an open set \( W_\gamma \subseteq Y \) to consist of

\[ \{[[\gamma \delta]] \in Y \mid \delta : I \to W \}. \]

Sets of the form \( W_\gamma \) make up a base for a topology on \( Y \); the verification of this is routine and tedious so we omit it.
7.5. The classification of covering spaces

With this topology, by restricting to open sets \( W \) that are locally simply connected we see that \( f \) has the property that it is a covering space map (in particular, it is continuous). Finally, \( f^{-1}(x) \) is isomorphic to \( K \setminus \pi_1(X, x_0) \cong S \) as a right \( \pi_1(X, x) \)-set, as required. \( \square \)

**Remark 7.53.** The hypotheses of Proposition 7.52 are slightly more restrictive than necessary. It is possible to prove it while assuming only that each \( x \in X \) has a local base of neighbourhoods \( U \) such that each loop in \( U \) based at \( x \) can be contracted to a point via a homotopy in \( X \)—our assumption is that the homotopy is defined in \( U \). An \( X \) satisfying the weaker condition is called *semilocally simply connected*. In practice, there are few examples we care about of spaces that are semilocally but not locally simply connected.

**Remark 7.54.** Suppose \( X \) is a path connected and locally simply connected space with basepoint \( x \). Among all the covering spaces of \( X \), there is one covering space (up to isomorphism) \( f : \tilde{X} \to X \) in particular that is very special: the one that has fibre isomorphic to \( \pi_1(X, x) \) itself as a set with right \( \pi_1(X, x) \)-action. This has the properties that:

1. \( \tilde{X} \) is path connected because \( \pi_1(X, x) \) consists of exactly one \( \pi_1(X, x) \)-orbit

2. The inclusion \( f_* : \pi_1(\tilde{X}, x) \to \pi_1(X, x) \) is the inclusion of the trivial group. This is because the fibre is isomorphic to \( \text{im}(f_*) \setminus \pi_1(X, x) \) as a set with right \( \pi_1(X, x) \)-action. In this case, this implies that \( \text{im}(f_*) \) is trivial.

Recall that a covering space \( f : \tilde{X} \to X \) with the property that \( \tilde{X} \) is simply connected is called the *universal cover* of \( X \). We have proved that if \( X \) is a path connected and locally simply connected space, then \( X \) has a universal cover. What is more, Corollary 7.51 shows that the universal cover is unique up to isomorphism of covering spaces.

**Remark 7.55.** What is universal about the universal cover? One might as well ask what is universal about the set \( \pi_1(X, x) \) as a set with right \( \pi_1(X, x) \)-action.

Suppose \( X \) is connected and locally path connected. Fix a universal cover \( f : \tilde{X} \to X \) and choose a basepoint \( \tilde{x} \in \tilde{X} \) (set \( x = f(\tilde{x}) \)). Given any covering space \( h : Y \to X \) and any point \( y \) in the fibre of \( h \) over \( x \), there exists a unique map of covering spaces \( \tilde{f} : \tilde{X} \to Y \) taking \( \tilde{x} \) to \( y \).

Showing this is an exercise in using Proposition 7.50 (to construct \( \tilde{f} \)) and Proposition 7.49 (to show it is unique).
7.6 Deck transformations

**Definition 7.56.** Let \( f : Y \to X \) be a covering space. An automorphism \( h : Y \to Y \) of \( Y \) over \( X \) is called a *deck transformation*. The set of all such transformations forms a group, \( \text{Aut}_X(Y) \).

**Remark 7.57.** The deck transformations are functions \( h : Y \to Y \), and therefore compose as functions do: \( h_2 \circ h_1 \) means “do \( h_1 \) and then do \( h_2 \)”. In particular, the group \( \text{Aut}_X(Y) \) is acting on \( Y \) on the left. This is notationally different from the right actions we have seen up until now, but practically very similar.

**Proposition 7.58.** Suppose \( X \) is a locally path connected space and \( f : Y \to X \) is a covering space. Let \( G \subset \text{Aut}_X(Y) \) be a subgroup (possibly the whole group). Then the action of \( G \) on \( Y \) is via a covering space action.

**Remark 7.59.** Next up, if \( X \) is connected and locally path connected, then \( f : Y \to X \) is entirely classified by the fibre \( F = f^{-1}(x) \) as a set with right \( \pi_1(X, x) \)-action. What’s more, \( \text{Aut}_X(Y) \) corresponds to the group of automorphisms of \( F \) as a set with right \( \pi_1(X, x) \)-action. Describing this group of automorphisms becomes entirely a question of group theory. To describe it, we need some terminology.

**Definition 7.60.** Let \( K \subseteq G \) be a subgroup of a group. Then \( \text{N}_G(K) \), the *normalizer* of \( K \) in \( G \) is the subgroup of elements \( g \in G \) such that \( gKg^{-1} = K \).

**Lemma 7.61.** Let \( K \subseteq G \) be a subgroup of a group. Then the group of automorphisms of \( K \setminus G \) as a right \( G \)-set is canonically identified with \( \text{N}_G(K)/K \).

**Proof.** The identification is by means of \( Kg \mapsto KnK \) where \( n \in \text{N}_G(K) \). \( \square \)

**Proposition 7.62.** Let \( X \) be connected and locally path connected. Let \( x_0 \in X \) be a basepoint and write \( G = \pi_1(X, x_0) \). Let \( f : Y \to X \) be a covering space such that \( Y \) is connected and has basepoint \( y_0 \) with \( f(y_0) = x_0 \). Write \( K \) for the image of the fundamental group of \( Y \) in \( G \). Then the group of deck transformations of \( Y \) over \( X \) is canonically identified with \( \text{N}_G(K)/K \).

**Proof.** Write \( G = \pi_1(X, x_0) \), and work in the equivalent category of right \( G \)-sets, by means of the functor \( F_{x_0} \). A deck transformation is then nothing more than an automorphism of \( F_{x_0}(Y) \) as a set with right \( G \)-action. \( \square \)

**Remark 7.63.** Retain the hypotheses imposed above on \( X \) and \( Y \). Since the deck transformations act on \( Y \) over \( X \), they act in particular on the fibre \( F = f^{-1}(x_0) \). This action is a left action, and is an action by \( \text{N}_G(K) \), which is, in general, a different group from \( G = \pi_1(X, x_0) \).
If we allow the covering space $Y$ to be disconnected, the group of deck transformations can become larger than $\pi_1(X, x_0)$. For instance, the split covering $* \sqcup * \to *$ has nontrivial deck transformation group.

In general, the left action by deck transformations is different from the right monodromy action by $\pi_1(X, x_0)$.

**Proposition 7.64.** Let $f : Y \to X$ be a covering space and let $H = N_G(K)/K$ be the group of deck transformations of $Y$ over $X$. Then the (left) action of $H$ on $Y$ is a covering space action.

**Definition 7.65.** Let $f : Y \to X$ be a covering space where $Y$ is connected and locally path connected. Let $y_0 \in Y$ be a basepoint mapping to $x_0 \in X$. We say $f : Y \to X$ is a normal covering space (for the basepoints $y_0$ and $x_0$) if the group of deck transformations $\text{Aut}_X(Y)$ acts transitively on $F_{x_0}$.

**Proposition 7.66.** Assume $f : Y \to X$ is a covering space where $Y$ is connected and locally path connected. The following are equivalent:

1. The covering is normal for one choice of basepoints $y_0 \in Y$ and $x_0 \in X$;
2. The subgroup $f_* (\pi_1(Y, y_0)) \subseteq \pi_1(X, x_0)$ is normal;
3. The covering is normal for all choices of basepoint.

**Proof.** The action by $N_G(K)/K$ is transitive on the fibre $K \setminus G$ if and only if $N_G(K) = G$, which is to say, if $K$ is normal in $G$.

The inclusion $f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is isomorphic to the inclusion $f_* : \pi_1(Y, y'_0) \to \pi_1(X, x'_0)$ given by a different choice of basepoints. \qed

**Remark 7.67.** Let $f : \tilde{X} \to X$ be a universal cover of a connected, locally path connected, semilocally simply connected space. Then $\text{Aut}_X(\tilde{X}) \cong \pi_1(X, x_0)$. 


Appendix A

Quotients and group actions

A.1 More about quotients

Recall that a function $q : X \to Y$ between topological spaces is a quotient map if $q$ is surjective and $Y$ has the finest topology such that $q$ is continuous. Equivalently, the topology on $Y$ is is such that $V \subseteq Y$ is open if and only if $q^{-1}(V)$ is open.

**Proposition A.1.** Suppose $q_1 : X \to Y$ and $q_2 : Y \to Z$ are quotient maps, then $q_2 \circ q_1$ is a quotient map.

**Definition A.2.** Let $q : X \to Y$ be a function. Say that a subset $U \subseteq X$ is saturated for $f$ if $U = q^{-1}(q(U))$.

**Remark A.3.** It is elementary to verify that if $V \subseteq Y$ is a subset, then $q^{-1}(V)$ is saturated. This is because $qq^{-1}(V) = V$ for any subset $V$ of $Y$.

Given any subset $U \subseteq X$, we can saturate $U$ by taking $q^{-1}q(U)$. If $U$ is not saturated, then $f^{-1}f(U)$ is strictly larger than $U$.

**Proposition A.4.** Let $q : X \to Y$ be a surjective (continuous) map. Then the following are equivalent:

1. $q : X \to Y$ is a quotient map;
2. For all saturated open sets $U \subseteq X$, the image $q(U)$ is open.

**Proof.** In one direction, suppose $q$ is a quotient map. Suppose $U \subseteq X$ is saturated and open. Then $q^{-1}q(U) = U$ is open in $X$, so $q(U)$ is open in $Y$.

In the other direction, we assume the images of saturated opens are open and wish to prove $q$ is a quotient map. Let $V \subseteq Y$ be a subset such that $q^{-1}(V)$ is open. The set
$q^{-1}(V)$ is saturated. By hypothesis, $qq^{-1}(V)$ is open, but $qq^{-1}(V) = V$. We have proved
that $V$ is open if $q^{-1}(V)$ is open. This proves that the map $q$ is a quotient map. \hfill $\square$

What follows is a surprisingly difficult theorem. It is due to J. H. C. Whitehead, from 1948.\footnote{The version he proved is slightly stronger}

**Theorem A.5.** Suppose $q : X \to Y$ be a quotient map and $Z$ is a locally compact Hausdorff
space. Then $q \times \text{id}_Z : X \times Z \to Y \times Z$ is a quotient map.

The hypothesis of local compactness cannot be dropped.

**Proof.** The map $q \times \text{id}_Z$ is continuous and surjective. It remains to show that $(q \times \text{id}_Z)(W)$
is open whenever $W$ is a saturated open set.

Let $W$ be a saturated open subset of $X \times Z$ and let $(y_0, z_0) \in (q \times \text{id}_Z)(W)$. We will
prove that $(q \times \text{id}_Z)(W)$ is open in $Y \times Z$ by constructing an open neighbourhood around
$(y_0, z_0)$ contained in $(q \times \text{id}_Z)(W)$.

Choose an arbitrary $x_0 \in q^{-1}(y_0)$. Then we can find a neighbourhood of $(x_0, z_0)$ of
the form $A \times B$, where $A \subset X$ is open and $B \subset Z$ is open. Because $Z$ is locally compact
and Hausdorff, we can further restrict to an open neighbourhood $V$ of $z_0$ such that $V \subset B$ and $V$ is compact. We concentrate on $V$.

Let $U$ denote the set

$$U = \{ x \in X \mid \{x\} \times \bar{V} \subset W \}.$$  

First, we observe that $x_0 \in U$: in fact $A \times \bar{V} \subset A \times B \subset W$.

Second, we claim that $U$ is saturated for the map $q : X \to Y$. Suppose $x \in U$ and
$x' \in X$ satisfy $q(x) = q(x')$. For all $z \in \bar{V}$, it is the case that $(x, z) \in W$. Since $W$ itself is
saturated, it is also the case that $(x', z) \in W$. Therefore $x' \in U$, establishing the claim.

Third, we show $U$ is open. Suppose $x \in U$. Then $\{x\} \times \bar{V}$ is a compact subset of
$X \times Z$ and $W$ is an open neighbourhood of this compact subset. By the generalized Tube
Lemma, we can find some open sets $D \subset X$ and $E \subset Z$ such that $\{x\} \times \bar{V} \subset D \times E \subset W$.
Choose any $x' \in D$ and any $z \in \bar{V}$, then $(x', z) \in W$. This shows that $x' \in U$, so that $D$
is an open neighbourhood of $x$ in $U$. This shows that $U$ is an open set.

We have therefore found two open sets $U \subset X$ and $V \subset Z$ such that

- $(x_0, z_0) \in U \times V$;
- $U$ is saturated for $q$;
- $U \times V \subset W$.\footnote{The version he proved is slightly stronger}
Apply \((q \times \text{id}_Z)(U \times V)\) to get \(q(U) \times Z\), which is an open subset of \(Y \times Z\) containing \((y_0, z_0)\) and contained in \((q \times \text{id}_Z)(W)\). Since \((y_0, z_0)\) was arbitrary, this shows that \((q \times \text{id}_Z)(W)\) is open, as required.

**Appendix A. Quotients and group actions**

**A.2 Topological Groups**

**Definition A.6.** A **topological group** is a space \(G\) equipped with a continuous multiplication map \(m : G \times G \to G\), an identity element \(e \in G\), and a continuous inversion \(i : G \to G\), all satisfying the usual axioms for a group.

**Example A.7.** Any group (not topological) can be given the discrete topology. This makes it a discrete group. This is the usual topology to put on a finite group.

**Example A.8.** The groups \(\text{GL}_n(\mathbb{R})\) and \(\text{GL}_n(\mathbb{C})\) are subsets of \(\mathbb{R}^{n^2}\) and \(\mathbb{C}^{n^2}\) respectively. If we give \(\text{GL}_n(\mathbb{R})\) and \(\text{GL}_n(\mathbb{C})\) the subspace topology, then they become topological groups.

**Remark A.9.** If \(G\) is a topological group and \(H\) is a subgroup of \(G\), then \(H\) inherits a topological group structure. The most common instances of this are when \(H\) is an open subgroup of \(G\) and, especially, when \(H\) is a closed subgroup of \(G\).

**Example A.10.** There are many closed subgroups of \(\text{GL}_n(\mathbb{R})\) and \(\text{GL}_n(\mathbb{C})\). They are all topological groups with the induced topology. Write \(A^T\) for the transpose of a matrix and \(A^*\) for the hermitian conjugate (i.e., the complex conjugate of the transpose of \(A\)). Each of the following determines a subgroup of \(\text{GL}_n(\mathbb{R})\) or \(\text{GL}_n(\mathbb{C})\) as appropriate:

1. \(O_n = \{A \in \text{GL}_n(\mathbb{R}) \mid AA^T = I_n\}\);
2. \(SO_n = \{A \in \text{GL}_n(\mathbb{R}) \mid AA^T = I_n, \ \det(A) = 1\}\);
3. \(U_n = \{A \in \text{GL}_n(\mathbb{C}) \mid AA^* = I_n\}\);
4. \(SU_n = \{A \in \text{GL}_n(\mathbb{C}) \mid AA^* = I_n, \ \det(A) = 1\}\).

They include \(O_n\) (the orthogonal groups), \(SO_n\) (the special orthogonal groups), \(U_n\) (unitary) and \(SU_n\) (special unitary).

**Example A.11.** The examples above include \(S^1\) in two different ways: \(S^1\) may be identified with \(SO_2\) (rotations of \(\mathbb{R}^2\)) and also with \(U_1\) (complex numbers of unit length).
A.3 Group actions

Definition A.12. Given a topological group $G$ and a space $X$, a left $G$-action on $X$ is a map $\alpha : G \times X \to X$ such that:

1. $\alpha(e, x) = x$ for all $x \in X$.
2. $\alpha(h, \alpha(g, x)) = \alpha(hg, x)$ for all $g, h \in G$ and $x \in X$.

Notation A.13. As well as $\alpha(g, x)$, different notations are used for group actions. For instance, $g \cdot x$ is common notation for $\alpha(g, x)$, or even just $gx$.

Remark A.14. A right $G$-action can be defined similarly. The key difference is in the composition rule. For a right action, $\alpha(h, \alpha(g, x)) = \alpha(gh, x)$. It is common to write right actions as $x \cdot g$ or $x^g$. Then you have the formulas $(x \cdot g) \cdot h = x \cdot (gh)$ or $(x^g)^h = x^{gh}$.

Remark A.15. A left $G$-action on $X$ can be converted into a right-$G$-action by letting $x \cdot g = g^{-1} \cdot x$. This means that theorems and definitions for left $G$-actions always have counterparts for right $G$-actions. As a matter of practice, it is best to be clear and unambiguous about whether one is using a left- or a right-$G$ action.

Remark A.16. All ordinary groups may be viewed as topological groups, by giving them the discrete topology. This means that everything we say here applies to ordinary (discrete) groups acting on topological spaces.

Definition A.17. If $\alpha : G \times X \to X$ is a left $G$-action and $x \in X$, then the stabilizer of $x \in X$ is the subgroup $G_x = \{ g \in G \mid g \cdot x = x \}$. A group action is called:

- **faithful** if the intersection of the stabilizers is trivial (in which case, every $g \in G \setminus \{e\}$ acts nontrivially on some element of $X$). For instance, $\text{GL}_n(\mathbb{R})$ acts faithfully on $\mathbb{R}^n$.

- **free** if the stabilizer of each $x \in X$ is trivial (in which case, every $g \in G \setminus \{e\}$ acts nontrivially on every element of $X$). For instance, a subgroup $H$ acts freely on $G$ by left-multiplication.

- **transitive** if, for every pair $(x, y)$ of elements in $X$, there is some $g \in G$ such that $gx = y$. In this case, $X$ consists of a single $G$-orbit.
A.4 Quotients by group actions

Construction A.18. Suppose $G \times X \to X$ is a left $G$-action. The notation $X/G$ denotes the set of equivalence classes of elements of $X$ under the relation $x \sim g \cdot x$. There is a surjective map $q : X \to X/G$. Give $X/G$ the quotient topology. We will call the resulting map $q : X \to X/G$ the quotient map and $X/G$ the quotient space.

Of course, a similar construction applies to right $G$-actions. In fact, the notation $X/G$ for quotients by left $G$-actions is unsatisfactory. It ought to be $G \backslash X$.

Proposition A.19. Suppose $G$ is topological group acting on a space $X$. Then the quotient map $q : X \to X/G$ is open.

Proof. Let $U$ be an open subset of $X$. Then $q^{-1}q(U) = \bigcup_{g \in G} g \cdot U$. Since $x \mapsto g \cdot x$ is a homeomorphism of $X$ with itself, the sets $g \cdot U$ are all open, hence $q^{-1}q(U)$ is open. This implies that $q(U)$ is open. \qed

Remark A.20. This contrasts with the case of general quotients in topology, which are not always open. For instance, the quotient map $q : [0, 1] \to S^1$ given by collapsing $\{0, 1\}$ is not an open map, as you can see by considering the image of $[0, 1/2)$.

Example A.21. Even if $G$ acts freely on a Hausdorff space $X$, the quotient space $X/G$ may not be Hausdorff. For example, consider the case of $\mathbb{Z}$ (with the discrete topology) acting on $\mathbb{R}^2 \setminus \{(0, 0)\}$ by $n \cdot (x, y) = (2^n x, 2^{-n} y)$.

Example A.22. A similar example to the above is that of $\mathbb{R}^\times$ acting on $\mathbb{R}^2 \setminus \{(0, 0)\}$ by $\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y)$.

Definition A.23. A map $f : X \to Y$ of topological spaces is proper if, for all compact sets $K \subset Y$, the subset $f^{-1}(K)$ is compact in $X$.

Remark A.24. The notion of proper map is most often used when both $X$ and $Y$ are locally compact and Hausdorff. With these additional hypotheses.
A.4. Quotients by group actions
Appendix B

The Compact–Open Topology

B.1 Definition

The fundamental problem in this section is how to put a topology on the set of all continuous functions $f : X \to Y$. In good circumstances, which is to say when $X$ is locally compact and Hausdorff, there is a standard default choice, which is the subject of this chapter.

The subject of functional analysis is the study of topologies and metrics on spaces of functions, and we know that functional analysis is not a small topic, so this chapter is not the last word on spaces of functions, by any means.

**Notation B.1.** Let $X$ and $Y$ be topological spaces. Let $K \subseteq X$ be compact and $U \subseteq Y$ be open. Write $o(K, U)$ for the set of continuous functions $f : X \to Y$ such that $f(K) \subseteq U$.

**Definition B.2.** Let $X$ and $Y$ be topological spaces. Let $\mathcal{C}(X, Y)$ denote the set of continuous functions from $X$ to $Y$, endowed with the topology generated by open sets of the form $o(K, U)$ as above. This topology is called the **compact–open** topology.

**Example B.3.** Suppose $K \subseteq X$ is a singleton $\{x\}$. This set is guaranteed to be compact. Let $U$ be any open set in $Y$. Then the set $o(\{x\}, U)$ is the set of continuous $f$ such that $f(x) \in U$.

This has the following consequence for the compact–open topology: if $(f_n) \to f$ is a convergent sequence in $\mathcal{C}(X, Y)$, and if $f(x) \in U$, then each open set $o(\{x\}, U)$ has to contain a tail of $(f_n)$. This is equivalent to saying that some tail $f_n(x), f_{n+1}(x), \ldots$ is contained in $U$, or in other other words, that $(f_n(x)) \to f(x)$.

**Example B.4.** Suppose $Y$ is a metric space, and $X$ is Hausdorff. In this case the compact-open topology is also called the **topology of uniform convergence on compact subsets** for the following reason.
A sequence of functions \((f_n)\) in \(\mathcal{C}(X, Y)\) converges to \(f\) in the compact-open topology if and only if, for all compact \(K \subseteq X\) and all \(\epsilon > 0\), there exists some \(N \in \mathbb{N}\) such that \(d(f_n(k), f(k)) < \epsilon\) for all \(k \in K\).

The “only if” direction is largely an exercise on the homework—the homework asks you to prove that when \(X = K\) is compact, the topology on \(\mathcal{C}(X, Y)\) is metric with the so-called uniform metric. Below we will also show that the map determined by restriction of functions \(\mathcal{C}(X, Y) \to \mathcal{C}(K, Y)\) is continuous.

Let’s do the “if” direction. Suppose \((f_n) \to f\) uniformly on all compact sets. Consider a general open neighbourhood of \(f\). This contains an open neighbourhood of the form

\[
W = \bigcap_{i=1}^{n} o(K_i, U_i) \ni f
\]

because opens like this form a basis.

**Proposition B.5.** The construction of \(\mathcal{C}(X, Y)\) is contravariantly functorial in \(X\) and covariantly functorial in \(Y\). In less technical language, if \(g : X' \to X\) and \(h : Y \to Y'\) are continuous functions, then precomposition with \(g\) and postcomposition with \(h\) yields a function

\[
\Phi_{g,h} : \mathcal{C}(X,Y) \to \mathcal{C}(X',Y') \quad f \mapsto h \circ f \circ g
\]

and this function is continuous.

**Proof.** It is sufficient to show that \(\Phi_{g,h}^{-1}(o(K, U))\) is open when \(K\) is compact in \(X'\) and \(U\) is open in \(Y'\). This is \(o(g(K'), h^{-1}(U))\), which is open. \(\square\)

**Proposition B.6.** Let \(Y\) be a topological space, then the map \(Y \to \mathcal{C}(*, Y)\) sending \(y\) to the constant function with value \(y\) is a homeomorphism.

### B.2 Currying and Uncurrying

Suppose we have a continuous function \(f : X \times Y \to Z\). We can *curry* this function to produce a function \(\alpha_f : X \to \mathcal{C}(Y, Z)\) defined by

\[
\alpha_f(x)(y) = f(x, y).
\]

The function \(\alpha_f(x) : Y \to Z\) is the composite

\[
Y \xrightarrow{i_x} X \times Y \xrightarrow{f} Z
\]

and since both functions here are continuous, so is \(\alpha_f(x)\).
Proposition B.7. The function \( \alpha_f : X \to \mathcal{C}(Y, Z) \) is continuous.

Proof. It suffices to prove that \( \alpha_f^{-1}(o(K, U)) \subseteq X \) is open, where \( K \subset Y \) is compact and \( U \subseteq Z \) is open.

Explicitly: \( \alpha_f^{-1}(o(K, U)) \) is the set of all \( x \in X \) such that for all \( y \in K \), the value \( f(x, y) \in U \).

Fix \( K, U \) and suppose \( x \in \alpha_f^{-1}(o(K, U)) \). This implies that \( f(\{x\} \times K) \subseteq U \) in \( Z \), or equivalently \( \{x\} \times K \subseteq f^{-1}(U) \). Then by the generalized tube lemma, there are some open \( V \ni x \) and \( W \supseteq K \) such that \( f(V \times W) \subseteq U \). In particular, \( f(V \times K) \subseteq U \), which implies that \( x \in V \subseteq \alpha_f^{-1}(o(K, U)) \). Since \( x \) was arbitrary, \( \alpha_f^{-1}(o(K, U)) \) is open. \( \square \)

We can also uncurry functions. Suppose \( f : X \to \mathcal{C}(Y, Z) \) is a continuous function, then we can define \( \beta_f : X \times Y \to Z \) by the formula \( \beta_f(x, y) = f(x)(y) \).

Proposition B.8. With the definition as above, if \( Y \) is locally compact and Hausdorff, then \( \beta_f \) is continuous.

Proof. Let \( U \) be an open set in \( Z \). We want to show that \( \beta_f^{-1}(U) \) is open. To do this, we take \( (x, y) \in \beta_f^{-1}(U) \) and show that it has some neighbourhood \( W \times V \) contained in \( \beta_f^{-1}(U) \).

The element \( f(x) \in \mathcal{C}(Y, Z) \) is a continuous function, so that \( f(x)^{-1}(U) \) is an open set of \( Y \) containing \( y \). Since \( Y \) is locally compact and Hausdorff, there is some open \( V \) satisfying \( y \in V \subseteq \tilde{V} \subseteq f(x)^{-1}(U) \) such that \( \tilde{V} \) is compact. Now consider the open set \( o(\tilde{V}, U) \) in \( \mathcal{C}(Y, Z) \). It contains \( f \). Furthermore, the set \( W = f^{-1}(o(\tilde{V}, U)) \) is open, because \( f \) is continuous, and it contains \( x \) because \( f(x)(\tilde{V}) \subseteq f(x)(f(x)^{-1}(U)) = U \).

Now if we apply \( \beta_f(W \times V) \), we get \( f(x')(y') \) where \( x' \in W \) and \( y' \in V \). Note that \( f(x') \in o(\tilde{V}, U) \), so that \( f(x')(y') \in U \), as required. \( \square \)

Corollary B.9. Let \( X, Z \) be topological spaces and \( Y \) a locally compact Hausdorff space. The two constructions of currying and uncurrying yield a natural bijective correspondence

\[
\mathcal{C}(X \times Y, Z) \leftrightarrow \mathcal{C}(X, \mathcal{C}(Y, Z))
\]

Remark B.10. If \( X \) and \( Y \) are both locally compact Hausdorff, then this is actually a homeomorphism.

Corollary B.11. Let \( X \) be a locally compact Hausdorff space. Then the evaluation function

\[
ev : \mathcal{C}(X, Y) \times X \to Y
\]

given by \( ev(f, x) = f(x) \) is continuous.
Proof. Apply the previous corollary to the bijection
\[ C(Y, Z) \times C(X, Y) \leftrightarrow C(C(Y, Z), C(X, Y)) \].

Take the identity function on the right. This corresponds to the evaluation function on the left. □

**Corollary B.12.** Let \( X \) and \( Z \) be topological spaces and let \( Y \) be a locally compact Hausdorff space. Then the composition function
\[ C(Y, Z) \times C(X, Y) \to C(X, Z) \]
is continuous.

Proof. It is equivalent to show the adjoint
\[ C(Y, Z) \times C(X, Y) \times X \to Z, \]
the map that sends \((g, f, x)\) to \(g \circ f(x)\), is continuous.

We can factor this as two evaluation maps
\[ C(Y, Z) \times C(X, Y) \times X \overset{id \times ev}{\to} C(Y, Z) \times Y \]
and
\[ C(Y, Z) \times Y \overset{ev}{\to} Z. \]
Both of these are continuous. □

### B.3 The pointed case
Appendix C

Category Theory

C.1 Categories, Functors and Natural Transformations

We generally disregard problems of size, viz. whether or not something is a set.

Definition C.1. A category $\mathbf{C}$ consists of a collection of objects, $\text{ob} \mathbf{C}$ and a collection of morphisms $\text{Mor} \mathbf{C}$, such that

1. Every morphism has a source in $\text{ob} \mathbf{C}$ and a target in $\text{ob} \mathbf{C}$. A morphism $f$ is often written $f : X \rightarrow Y$ or $X \xrightarrow{f} Y$, where $X$ is the source and $Y$ is the target.

2. For any two objects $X$ and $Y$, there is a set $\text{Mor}_\mathbf{C}(X, Y)$ or $\mathbf{C}(X, Y)$, consisting of precisely those morphisms of $\mathbf{C}$ having source $X$ and target $Y$.

3. For any three objects $X, Y, Z$ of $\mathbf{C}$, there is a composition of morphisms

$$\circ : \mathbf{C}(X, Y) \times \mathbf{C}(Y, Z) \rightarrow \mathbf{C}(X, Z)$$

and this composition is associative in that $f \circ (g \circ h) = (f \circ g) \circ h$ whenever these composites are defined.

4. For each object $X$ of $\mathbf{C}$, there exists an identity morphism $\text{id}_X \in \mathbf{C}(X, X)$ such that $f \circ \text{id}_X = f$ and $\text{id}_X \circ g = g$ whenever these composites are defined.

Remark C.2. An easy and standard argument proves that $\text{id}_X$ is the unique morphism $X \rightarrow X$ with the stated property.

Notation C.3. There are categories $\text{Set}, \text{Gr}, \text{Ab}$, of sets, groups, abelian groups, and many other similar categories of objects commonly studied in mathematics. These are generally large categories, in that the collection of objects does not form a set.
Example C.4. There are also *small* categories, where the collection of objects forms a set, and therefore the collection of morphisms also forms a set (under our hypotheses). For instance, given any partially ordered set $S$, one can construct a category, also called $S$, where one regards ‘element of’ and ‘object of’ as synonymous, and then declares that $S(a, b) = \emptyset$ if $b < a$ and that $S(a, b)$ consists of one morphism if $a \leq b$.

It is often possible to depict such small categories diagrammatically. It is customary to draw only a subset of all morphisms, and to leave out morphisms that can be inferred from the morphisms and objects drawn. In particular, identity morphisms are seldom drawn.

1. The standard span:
   $$
   \bullet \leftarrow \bullet \rightarrow \bullet
   $$

2. The standard cospan:
   $$
   \bullet \rightarrow \bullet \leftarrow \bullet
   $$

3. The category $\mathbb{N}$ (with the usual order)
   $$
   0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots
   $$

Example C.5. There is a category $\textbf{Top}$ of topological spaces where the objects are topological spaces and morphisms are continuous functions. There is also a category of *pointed spaces*, $\textbf{Top}_*$, where the objects are pairs $(X, x_0)$ where $X$ is a topological space and $x_0 \in X$. The morphisms are the based maps, i.e., $\text{Hom}_*((X, x_0), (Y, y_0))$ is the set of continuous $f : X \to Y$ such that $f(x_0) = y_0$.

Definition C.6. Given a category $\mathcal{C}$, a *subcategory* $\mathcal{D}$ of $\mathcal{C}$ consists of a subcollection $\text{ob} \mathcal{D}$ of $\text{ob} \mathcal{C}$ and a subcollection $\text{Mor} \mathcal{D}$ of $\text{Mor} \mathcal{C}$, containing $\text{id}_X$ for all objects $X$ in $\text{ob} \mathcal{D}$, such that $\text{Mor} \mathcal{D}$ is closed under composition.

Example C.7. There are many examples of subcategories that arise by restricting the class of objects, but not restricting the morphisms between the objects. For instance, $\text{Ab}$ is the subcategory of $\textbf{Gr}$ where the groups considered are required to be abelian, but given any two abelian groups $G, H$, one has $\text{Gr}(G, H) = \text{Ab}(G, H)$. In this situation, $\text{Ab}$ is a *full* subcategory of $\textbf{Gr}$.

Example C.8. At the other extreme, it is possible to form subcategories where one considers all the objects, but strictly fewer morphisms. For instance, given a field $k$, one might consider the category having as objects the collection of finite-dimensional $k$ vector spaces, but where the morphisms are restricted to be isomorphisms. This is a subcategory of the usual category of finite-dimensional $k$ vector spaces and all $k$ linear maps, and it appears in some definitions of algebraic $K$-theory.
**Definition C.9.** Given two categories, \( C \) and \( D \), it is possible to form a product category \( C \times D \). The objects in this category are ordered pairs \((X, Y)\) where \( X \) is an object of \( C \) and \( Y \) is an object of \( D \). The morphisms are also ordered pairs, \((f, g) : (X, Y) \to (Z, W)\) is a morphism in the product category if \( f : X \to Z \) is a morphism in \( C \) and \( g : Y \to W \) is a morphism in \( D \).

**Definition C.10.** If \( C \) is a category, and \( f : X \to Y \) is a morphism in this category, then we say that \( f \) is an isomorphism if there exists a morphism \( f^{-1} : Y \to X \) such that \( f^{-1} \circ f = \text{id}_X \) and \( f \circ f^{-1} = \text{id}_Y \). It is immediate that \( \text{id}_X \) is an isomorphism.

**Remark C.11.** An isomorphism in \( \Pi \) or a related category is generally called a homeomorphism.

**Definition C.12.** If \( C \) is a category, and \( f : X \to Y \) is a morphism in this category, then we say that \( f \) is

1. a monomorphism if, whenever \( g, h : Z \to X \) are morphisms, the statement \( f \circ g = f \circ h \) implies \( g = h \). That is, \( f \) is left cancellable,

2. an epimorphism if, whenever \( g, h : Y \to Z \) are morphisms, the statement \( g \circ f = h \circ f \) implies \( g = h \). That is, \( f \) is right cancellable,

3. a bimorphism if it is both a monomorphism and an epimorphism.

**Definition C.13.** If \( C \) is a category, and \( f : X \to Y \) is a morphism in this category, then we say that \( f \) is

1. a split monomorphism if there exists a morphism \( g : Y \to X \) such that \( g \circ f = \text{id}_X \).

2. a split epimorphism if there exists a morphism \( g : Y \to X \) such that \( f \circ g = \text{id}_Y \).

**Exercises**

1. Suppose \( f : X \to Y \) is an isomorphism. Prove that \( f^{-1} \) is uniquely determined by \( f \).

2. Prove that the class of isomorphisms in a category has the two-out-of-three property, namely: if

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\end{array}
\]

are composable morphisms such that two of \( f, g \) and \( g \circ f \) are isomorphisms, then so too is the third.
3. Prove that the class of isomorphisms in a category has the two-out-of-six property, namely: if

\[ A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} F \]

are composable morphisms such that \( g \circ f \) and \( h \circ g \) are isomorphisms, then so too are \( f, g, h \) and \( h \circ g \circ f \).

4. Determine the monomorphisms, epimorphisms and bimorphisms in the category of sets.

5. Give an example in \( \text{Top} \) of a bimorphism that is not an isomorphism.

6. Let \( \text{Haus} \) denote the full subcategory of Hausdorff topological spaces. Give an example in \( \text{Haus} \) of an epimorphism that is not surjective.

C.2 Functors and Natural Transformations

Definition C.14. Given two categories \( C \) and \( D \), a (covariant) functor \( F : C \to D \) consists of an assignment

\[ F : \text{ob } C \to \text{ob } D \]

and for every pair of objects \( X, Y \) in \( \text{ob } C \), a function

\[ F : C(X, Y) \to D(F(X), F(Y)) \]

such that

1. \( F(\text{id}_X) = \text{id}_{F(X)} \) for all object \( X \) of \( C \) and

2. \( F(f \circ g) = F(f) \circ F(g) \) wherever \( f \circ g \) is defined.

Example C.15. Given any category \( C \), there is an identity functor \( \text{id}_C \).

Definition C.16. Given two categories \( C \) and \( D \), a contravariant functor \( F : C \to D \) consists of an assignment

\[ F : \text{ob } C \to \text{ob } D \]

and for every pair of objects \( X, Y \) in \( \text{ob } C \), a function

\[ F : C(X, Y) \to D(F(Y), F(X)) \]

such that
1. $F(\text{id}_X) = \text{id}_{F(X)}$ for all object $X$ of $C$ and

2. $F(f \circ g) = F(g) \circ F(f)$ wherever $f \circ g$ is defined.

Remark C.17. Warning: contravariant functors reverse the direction of morphisms. Failure to keep adequate track of the variance of functors is the category-theoretical analogue of a sign error in arithmetic. These errors are minor, frustrating and common.

Notation C.18. Given a category $C$, there is an opposite category, $C^{\text{op}}$ having the same collection of objects, but where

$$C^{\text{op}}(X, Y) = C(Y, X).$$

One may view a contravariant functor $F : C \to D$ as a covariant functor $F : C^{\text{op}} \to D$.

Example C.19. There are many functors in mathematics that consist largely of forgetting structures. Such functors are often called “forgetful”, but it is difficult to give a precise definition of what this means. Common examples include:

1. $V : \Pi_* \to \Pi$, forgetting the basepoint.

2. $V : \Pi \to \text{Set}$, forgetting the topology.

3. $V : \text{Ab} \to \text{Grp}$, forgetting that the group is abelian.

Example C.20. There is a canonical functor $\eta : C^{\text{op}} \times C \to \text{Set}$ given by $\eta(X, Y) = C(X, Y)$. Fixing either $X$ or $Y$ gives rise to functors

1. $\eta_X : C \to \text{Set},$

2. $\eta^Y : C^{\text{op}} \to \text{Set}.$

Definition C.21. Let $F : C \to D$ be a functor. We say $F$ is

1. full if, for any two objects $X, Y$ of $C$, the function $F : C(X, Y) \to D(F(X), F(Y))$ is surjective.

2. faithful if, for any two objects $X, Y$ of $C$, the function $F : C(X, Y) \to D(F(X), F(Y))$ is injective.

3. essentially surjective if, for any object $Z$ of $D$, one can find an object $X$ of $C$ such that there exists an isomorphism $Z \to F(X).$
Definition C.22. Given two (covariant) functors $F, G : C \to D$, a natural transformation $\Psi : F \to G$ consists of a collection of morphisms $\Psi_X : F(X) \to G(X)$, one for each object $X$ of $C$, such that for any morphism $h : X \to Y$ in the category $C$, the square

\[
\begin{array}{ccc}
F(X) & \xrightarrow{\Psi_X} & G(X) \\
\downarrow^{F(h)} & & \downarrow^{G(h)} \\
F(Y) & \xrightarrow{\Psi_Y} & G(Y)
\end{array}
\]

commutes, which is to say: $G(h) \circ \Psi_X = \Psi_Y \circ F(h)$.

Remark C.23. A similar definition of natural transformation can be made if $F$ and $G$ are both contravariant. The details are left to the reader.

The word “natural” is often applied to morphisms between objects in categories. It should be used only to apply to a morphism that is part of a, possibly implicit, natural transformation. If the morphisms $\Psi_X$ are all of a certain type, for instance all isomorphisms or all inclusions, then $\Psi_X$ may be said to be a natural isomorphism or a natural inclusion as appropriate.

Example C.24. Fix a field $k$. Let $k\text{Vect}$ denote the category of $k$ vector spaces and all linear maps between them. Then there is a contravariant functor sending $f : V \to W$ to $f^* : W^* \to V^*$, where $V^* = \text{Hom}_k(V, k)$ and $f^*$ is the evident map $\text{Hom}_k(W, k) \to \text{Hom}_k(V, k)$ given by postcomposing with $f$.

There is a covariant functor sending $f : V \to W$ to $f^{**} : V^{**} \to W^{**}$ given by applying $V^*$ twice. That is, $V^{**}$ is the $k$ vector space of linear functionals on the $k$ vector space of linear functionals on $V$. There is a natural transformation $e : \text{id}_{k\text{Vect}} \to (\cdot)^{**}$ given by a collection of $k$-linear maps $e_V : V \to V^{**}$ given by defining $e_V(x)$, where $x \in V$, to be the functional sending $\psi \in V^*$ to $\psi(x)$.

At least if one assumes the Axiom of Choice, the map $e_V : V \to V^{**}$ defined above is a natural inclusion. If one restricts to the full subcategory of finite dimensional $k$ vector spaces, then $e$ is a natural isomorphism, but if $V$ is not finite dimensional, then $e_V : V \to V^{**}$ is not an isomorphism.

Definition C.25. If $F : C \to D$ is a functor, then we say $F$ is an equivalence of categories if there exists a functor $G : D \to C$ and natural isomorphisms $\Phi : G \circ F \to \text{id}_C$ and $\Psi : F \circ G \to \text{id}_D$.

Remark C.26. In contrast to the case of isomorphisms, the functor $F$ is not sufficient to determine $G$, $\Psi$ and $\Phi$ uniquely. The notion of “isomorphism of categories”, where $G \circ F$ and $F \circ G$ are required to be identity functors, is not particularly common.
Remark C.27. In the presence of a sufficiently strong version of the Axiom of Choice, a functor is an equivalence of categories if an and only if it is full, faithful, and essentially surjective.

Example C.28. Let \( \text{Fin} \) denote the category of finite sets. This category is not small. Let \( \mathbf{N} \) denote the full subcategory of sets \( \{\emptyset, \{1\}, \{1, 2\}, \ldots\} \). Then \( \mathbf{N} \to \text{Fin} \) is an equivalence of categories. In this situation, one says that \( \mathbf{N} \) is a small skeleton for \( \text{Fin} \).

Remark C.29. If one restricts attention to small categories, then one can define a “category of categories”, but as we have remarked, the notion of isomorphism one gets is not generally useful. It is better to incorporate the natural transformations and form a “2-category” of small categories, a structure having objects (categories), morphisms (functors), and morphisms of morphisms (natural transformations). We will not pursue this further here.

Exercises

1. Let \( F : C \to D \) be a functor. Show that \( F \) preserves isomorphisms and split mono- and epimorphisms. Show by example that it need not preserve monomorphisms or epimorphisms that are not split.

C.3 Adjoint Functors

Definition C.30. Given two functors \( L : C \to D \) and \( R : D \to C \), we say \( L \) is left adjoint to \( R \) and \( R \) is right adjoint to \( L \) if, for any object \( X \) of \( C \) and \( Y \) of \( D \), there exists a bijection

\[
\Psi_{X,Y} : D(L(X), Y) \to C(X, R(Y))
\]

and such that the bijection \( \Psi \) is a natural isomorphism of functors \( C^{\text{op}} \times D \to \text{Set} \). More explicitly, if \( f : X \to X' \) and \( g : Y' \to Y \) are morphisms in the appropriate categories, then the square of sets

\[
\begin{array}{ccc}
D(L(X'), Y') & \xrightarrow{\Psi_{X',Y'}} & C(X', R(Y')) \\
\downarrow & & \downarrow \\
D(L(X), Y) & \xrightarrow{\Psi_{X,Y}} & C(X, R(Y))
\end{array}
\]

commutes.

Example C.31. Forgetful functors often have one or both kinds of adjoint. For instance, the forgetful functor \( V : \text{Top} \to \text{Set} \) has both a left- and a right-adjoint. The forgetful functor \( V : \text{Ab} \to \text{Set} \) has a left adjoint, but no right adjoint.
Example C.32. A very important family of adjunctions is modelled on the following one: fix a set $X$. This gives rise to two functors $\text{Set} \to \text{Set}$; the cartesian product functor $Y \mapsto Y \times X$, and the mapping space functor $Z \mapsto Z^X$, where $Z^X$ is notation for the set of functions $X \to Z$. That these are indeed functors $\text{Set} \to \text{Set}$ is left as an exercise. We assert that they form an adjoint pair, in that there is a natural bijection

$$\text{Set}(Y \times X, Z) \to \text{Set}(Y, Z^X).$$

Verifying this is left to the reader.

Example C.33. The previous example has a variant for topological spaces, provided some additional hypothesis is placed on the spaces appearing. For instance, if $X$ is a locally compact Hausdorff space, then there is a natural bijection

$$\text{Top}(Y \times X, Z) \to \text{Top}(Y, C(X, Z))$$

where $C(X, Z)$ is the space of continuous functions $X \to Z$ given the compact–open topology.

Definition C.34. Given an adjoint pair of functors $L : C \to D$ and $R : D \to C$, we can define two natural transformations.

1. The unit of the adjunction $\epsilon : \text{id}_C \to R \circ L$

2. The counit of the adjunction $\eta : L \circ R \to \text{id}_D$.

The unit is formed by letting $\eta_X : X \to R(L(X))$ be the element of $C(X, R(L(X)))$ corresponding to $\text{id}_{L(X)} \in D(L(X), L(X))$ under the natural isomorphism of the adjunction. The counit is formed similarly.

Remark C.35. We continue with the notation of the previous definition. The unit and counit have certain universal properties. In the case of the unit, suppose that there is a morphism $f : X \to R(Y)$ in $C$. Since $L$ and $R$ are adjoint, the morphism $f$ is equivalent to a unique morphism $g : L(X) \to Y$. This morphism can be written, tautologically, as $\text{id}_{L(X)} \circ g : L(X) \to L(X) \to Y$, which, by adjunction, is equivalent to a factorization $f = R(g) \circ \epsilon_X : X \to R(L(X)) \to R(Y)$.

Dually, any morphism $h : L(X) \to Y$ factors uniquely as $\eta_Y \circ L(i) : L(X) \to L(R(Y)) \to Y$.

Remark C.36. If $L : C \to D$ and $M : D \to E$ are two functors, each left adjoint to functors $R$ and $S$ respectively, then $M \circ L$ is left adjoint to $R \circ S$.

Proposition C.37. Suppose $L, L' : C \to D$ are two naturally isomorphic functors and $R, R'$ are right adjoints to $L$ and $L'$. Then $R$ and $R'$ are naturally isomorphic.

This result applies in particular in the case where $L = L'$. 
C.4 Diagrams, Limits and Colimits

**Notation C.38.** If $I$ is a small category and $C$ is a category, then a functor $D : I \to C$ may be called a *diagram*. If, for any morphism $f : i \to j$ in the category $I$, the morphism $D(f)$ depends only on $i$ and $j$, then we say the diagram is *commutative*.

**Example C.39.** Not all commonly occurring diagrams are commutative. For instance, pairs of parallel morphisms $X \Rightarrow Y$ appear often but form a commutative diagram only when the two morphisms agree.

**Definition C.40.** Given a small category $I$ and a category $C$, one can define a category $\text{Fun}(I, C)$ of $I$-shaped diagrams. The objects are the functors $D : I \to C$, and the morphisms are the natural transformations between them.

**Definition C.41.** Given a small category $I$, a category $C$ and an object $X$ of $C$, we can form the *constant $I$-shaped diagram with value $X$* by $\text{const}_I(X) : I \to C$ by sending all objects to $X$ and all morphisms to $\text{id}_X$. In fact, $\text{const}_I$ is a functor $\text{const}_I : C \to \text{Fun}(I, C)$.

**Definition C.42.** Let $I$ be a small category and $C$ a category.

Given an $I$-shaped diagram $D$ in $C$, a *limit* of $D$ is an object $\lim D$ of $C$ and a natural transformation $\Phi : \text{const}_I(\lim D) \to D$ such that for any object $X$ of $C$ equipped with a natural transformation $\Psi : \text{const}_I(X) \to D$, there is a unique map $u : X \to \lim D$ such that $\Psi = \Phi \circ \text{const}(u)$.

Dually, a *colimit* of an $I$-shaped diagram $D$ is an object $\text{colim} D$ of $C$ and a natural transformation $\Phi : D \to \text{const}_I(\text{colim} D)$ such that for any object $X$ of $C$ equipped with a natural transformation $\Psi : D \to \text{const}_I(X)$, there is a unique map $u : \text{colim} D \to X$ such that $\Psi = \text{const}(u) \circ \Phi$.

**Remark C.43.** Strictly speaking a limit or colimit of a diagram encompasses both the object and the natural transformation of functors—which is to say, the morphisms. In practice, one often refers to the object as the limit or colimit, leaving the morphisms implicit.

**Remark C.44.** It follows easily from a standard argument that if $L$ and $L'$ are two limits of the same diagram $D : I \to C$, then there is a unique isomorphism $f : L \to L'$ in $C$ such that the diagram

$$
\begin{array}{ccc}
\text{const}_I L & \xrightarrow{\text{const}(f)} & \text{const}_I L' \\
D & \searrow & \\
& \text{const}_I &
\end{array}
$$

commutes. A dual statement applies to colimits.
Since they are unique up to unique isomorphism, one often abuses terminology and speaks of “the limit” or “the colimit” of a diagram.

**Remark C.45.** There is another view of limits and colimits that is sometimes useful. Suppose the functor const\(_I\) has a right adjoint \(\ell\). Then a limit of \(D\) is given by the object \(\ell(D)\) and the counit map \(\text{const}_I \ell(D) \to D\).

Dually, if \(\text{const}_I\) has a right adjoint \(\text{colim}\), the colimit of \(D\) is the unit map \(D \to \text{const}_I \text{colim}(D)\).

**Example C.46.** The language used above is technical. In practice, the idea is simple. Let us consider as a category \(I\) the standard cospan

\[
\bullet \longrightarrow \bullet \leftarrow \bullet
\]

Let \(C = \text{Top}\) be the category of topological spaces. Then the data of an \(I\)-shaped diagram \(D\) consists of three spaces and two continuous functions \(X \to Y \leftarrow Z\).

The constant-diagram functor takes a space \(W\) and produces \(W \to W \leftarrow W\), where the morphisms are identities. A natural transformation \(\text{const}(W) \to D\) is the data of continuous functions \(f : W \to X, g : W \to Y\) and \(h : W \to Z\) such that

\[
\begin{array}{ccc}
W & \longrightarrow & W \\
\downarrow f & & \downarrow g \\
X & \longrightarrow & Y \\
& & \downarrow h \\
& & \leftarrow Z
\end{array}
\]

commutes, or, more succinctly

\[\text{(C.1)}\]

Note further that the dotted arrow is determined by either \(f\) or \(h\), and may be omitted.

The space \(\text{lim } D\) and the natural transformation amounts to an object and morphisms fitting in the following diagram

\[
\text{lim } D \xrightarrow{h} X \\
\downarrow f \\
Y \longrightarrow Z.
\]

\[\text{(C.2)}\]
This diagram has the property that if $W$ is as in Diagram (C.1), then there exists a unique map $W \to \lim D$ such that Diagram (C.3) commutes.

This particular kind of limit is called a fibre product and is written $X \times_Y Z$. While our definition specifies the limit only up to unique isomorphism, we can easily construct an explicit model for $X \times_Y Z$ in the category of topological spaces. Most usually, let $X \times_Y Z$ consist of the subset of pairs $(x, z) \in X \times Z$ such that the image of $x$ and of $z$ in $Y$ agree. Then endow $X \times_Y Z$ with the coarsest topology (fewest open sets) such that the evident projection maps $X \times_Y Z \to X$ and $X \times_Y Z \to Z$ are both continuous.

It is instructive to consider $X \times_Y Z$ in the following cases:

1. When $Y$ is a singleton space.
2. When $X \to Y$ is the inclusion of a subspace.

Remark C.47. By uniqueness of adjoints and of unit or counit maps, if a limit or colimit of a diagram exists, it is unique up to unique isomorphism.

Notation C.48. A category in which all limits can be constructed is complete and one in which all colimits can be constructed is cocomplete. The following categories are all complete and cocomplete:

1. Set.
2. Top and Top$_*$. 
3. $R$-Mod.

The full subcategory Haus of Hausdorff topological spaces is complete but not cocomplete.

Notation C.49. If $D$ is a diagram in C consisting of a family of objects $\{X_i\}_{i \in I}$ and no nonidentity arrows, then a limit of $D$ is called a product of $\{X_i\}_{i \in I}$ and a colimit of $D$ is called a coproduct of $\{X_i\}_{i \in I}$. The product of topological spaces is an example of a categorical product, and the disjoint union of topological spaces is an example of a categorical coproduct.
Notation C.50. If \( D \) is a diagram in \( C \) of the form
\[
\begin{array}{ccc}
A & \downarrow & \\
B & \rightarrow & C
\end{array}
\]
then a limit of \( D \) is called a pullback of \( D \), and often denoted \( A \times_C B \).

The dual concept is the pushout, a colimit of.
\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \\
C & &
\end{array}
\]

Proposition C.51. Suppose \( F : C \to C \) is a functor between complete categories such that \( F \) has a left adjoint, \( L \). Suppose further that \( D \) is a diagram in \( C \). Let \( \text{lim} \, D \) be a limit of \( D \). Then \( F(\text{lim} \, D) \) is a limit of \( F(D) \).

Dually, suppose \( F : C \to C \) is a functor between cocomplete categories such that \( F \) has a right adjoint, \( R \). Suppose further that \( D \) is a diagram in \( C \). Let \( \text{colim} \, D \) be a limit of \( D \). Then \( F(\text{colim} \, D) \) is a colimit of \( F(D) \).

Remark C.52. Let \( C \) be a category. Consider the empty diagram \( D \). If \( \text{lim} \, D \) exists, then it is an object \( * \) such that all objects \( X \) of \( C \) are equipped with a unique morphism \( X \to * \). Such an object \( * \) is called a terminal object of \( C \). Any two terminal objects are isomorphic by a unique isomorphism.

Dually, the colimit of an empty diagram is called an initial object; such an object may often be denoted \( \emptyset \). If an object is both initial and terminal, then it is called a zero object.

Exercises

1. The forgetful functor \( V : \text{Ab} \to \text{Set} \) has a left adjoint, \( L \). Describe the unit map \( \epsilon : S \to V(L(S)) \).

2. Show that \( V : \text{Ab} \to \text{Set} \) does not preserve colimits. For instance, consider the colimit of a diagram consisting of two nonzero abelian groups and no nontrivial arrows. Therefore \( V \) does not have a right adjoint.
3. Let $R$ be a ring and let $\mathbf{M}$ denote the category of $R$-modules and $R$-linear maps, and let $f : M \to N$ be a morphism in $\mathbf{M}$. Describe the limit of the diagram

\[
\begin{array}{ccc}
& & 0 \\
& \downarrow & \\
M & \xrightarrow{f} & N
\end{array}
\]

Express the cokernel of $f$ as the colimit of a diagram.

4. Consider the forgetful functor $V : \mathbf{Top}_\bullet \to \mathbf{Top}$. Describe a left adjoint to this functor. Prove that $V$ does not have a right adjoint.

5. Let $X$ be a locally compact Hausdorff space, and consider the adjunction between $\times X$ and $\mathcal{C}(X, -)$ in $\mathbf{Top}$. Describe the counit of this adjunction.
Appendix D

\( p \)-Norms

In this appendix we assume an extended real line, where \( \infty \) is an element greater than all real numbers; the interval notation \([1, \infty]\) will be used to mean \([1, \infty) \cup \{\infty}\).

D.1 The \( p \) norms on \( \mathbb{R}^n \)

Fix an integer \( n \geq 1 \). When \( p \geq 1 \) is a real number, we define

\[
\|\vec{x}\|_p = \left( \sum_{i=0}^{n} |x_i|^p \right)^{1/p}.
\]

Define

\[\|\vec{x}\|_\infty = \sup_i |x_i|.
\]

Each of these norms has the following property: given any \( \vec{x} \in \mathbb{R}^n \) and any \( r \in \mathbb{R} \), we have

\[
\|r\vec{x}\|_p = |r|\|\vec{x}\|_p.
\]

This will be important later. It is also immediate, for \( p \in [1, \infty] \), that \( \|\vec{x}\|_p = 0 \) if and only if \( \vec{x} = 0 \).

Hölder conjugates

For a given real number \( p > 1 \), the Hölder conjugate of \( p \) is the number \( q > 1 \) such that

\[
\frac{1}{p} + \frac{1}{q} = 1;
\]
this is equivalent to
\[ q = \frac{p}{p-1}. \tag{D.2} \]

Another equivalent formulation is
\[ qp - q - p = 0 \tag{D.3} \]

Observe that 2 is self-conjugate, but no other number is. We also declare the pair \( \{1, \infty\} \) to be Hölder conjugates.

**Proposition D.1** (Young’s Inequality). Let \( p, q \) be a Hölder conjugate pair in \((1, \infty)\) and suppose \( a, b \) are nonnegative real numbers, then
\[
a b \leq \frac{a^p}{p} + \frac{b^q}{q}
\]
with equality if and only if \( a^p = b^q \).

**Exercise D.2.** Define
\[
f(x) = \frac{x^p}{p} + \frac{b^q}{q} - bx.
\]
Using calculus, prove this function has a unique global minimum on \((0, \infty)\) and find that minimum.

**Proposition D.3** (Hölder’s Inequality). For a given \( p \in [1, \infty] \), having Hölder conjugate \( q \), and any two vectors \( \vec{x}, \vec{y} \) in \( \mathbb{R}^n \), one has
\[
\sum_{i=1}^{n} |x_i y_i| \leq \|\vec{x}\|_p \|\vec{y}\|_q.
\]

**Proof.** When \( p = 1 \) and \( q = \infty \), or vice versa, this amounts to the triangle inequality for the absolute value on \( \mathbb{R}^1 \).

We therefore assume \( 1 < p < \infty \). By referring to (D.1), we see that it suffices to prove the proposition after replacing \( \vec{x} \) and \( \vec{y} \) by \( r \vec{x} \) and \( s \vec{y} \) where \( 0 < r \) and \( 0 < s \), so we may assume that \( \|\vec{x}\|_p = \|\vec{y}\|_q = 1 \).

By repeated use of Young’s inequality, we obtain the inequality
\[
\sum_{i=1}^{n} |x_i y_i| \leq \sum_{i=1}^{n} \left( \frac{|x_i|^p}{p} + \frac{|y_i|^q}{q} \right),
\]
which is
\[ \sum_{i=1}^{n} |x_i y_i| \leq \frac{\sum_{i=1}^{n} |x_i|^{p}}{p} + \frac{\sum_{i=1}^{n} |y_i|^{q}}{q} = \frac{\|\mathbf{x}\|_p^p}{p} + \frac{\|\mathbf{y}\|_q^q}{q} = 1 = \|\mathbf{x}\|_p \|\mathbf{y}\|_q. \]

□

**Proposition D.4** (Minkowski Inequality). Let \( p \in [1, \infty] \) and \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \), then
\[ \|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p. \]

**Proof.** The cases of \( p = 1 \) and \( p = \infty \) reduce immediately to the usual triangle inequality for \( \mathbb{R} \).

Assume that \( 1 < p < \infty \).

Consider a vector \( \mathbf{w} \) having \( i \)-th coordinate \( w_i = |x_i + y_i|^{p-1} \). We calculate
\[ \|\mathbf{w}\|_q = \left( \sum_{i=1}^{n} |x_i + y_i|^{q(p-1)} \right)^{1/q} = \left( \sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/q} = \|\mathbf{x} + \mathbf{y}\|_p^{p/q}. \]

Now we split up \( \|\mathbf{x} + \mathbf{y}\|_p^p \) as follows:
\[ \|\mathbf{x} + \mathbf{y}\|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p \leq \sum_{i=1}^{n} \left( |x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1} \right) \leq \|\mathbf{x}\|_p \|\mathbf{w}\|_q + \|\mathbf{y}\|_p \|\mathbf{w}\|_q \]

\[ \frac{\|\mathbf{w}\|_q}{p} \]

where the last inequality is the Hölder inequality. We have a formula for \( \|\mathbf{w}\|_q \), which we use to deduce
\[ \|\mathbf{x} + \mathbf{y}\|_p^p \leq \|\mathbf{x}\|_p \|\mathbf{w}\|_q + \|\mathbf{y}\|_p \|\mathbf{w}\|_q = (\|\mathbf{x}\|_p + \|\mathbf{y}\|_p) \|\mathbf{x} + \mathbf{y}\|_p^{p/q}. \]

Dividing through gives
\[ \|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p, \]

which is what we wanted.

**Exercise D.5.** Show that for a given vector \( \mathbf{x} \in \mathbb{R}^n \), the function \( p \mapsto \|\mathbf{x}\|_p \) is decreasing on \( p \in [1, \infty] \).

Show further that \( \lim_{p \to \infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty. \)

**Proposition D.6.** Given \( \mathbf{x} \in \mathbb{R}^n \) and any \( p \in [1, \infty] \)
\[ \|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_p \geq \|\mathbf{x}\|_\infty \geq \frac{1}{n} \|\mathbf{x}\|_1. \]
This follows from Exercise D.5 and the observation that \( \|\vec{x}\|_1 \leq n\|\vec{x}\|_\infty \).

**Proposition D.7.** Given any \( \{p, q\} \subset [1, \infty] \), there exist constants \( c, C > 0 \), such that for any \( \vec{x} \in \mathbb{R}^n \), we have
\[
C\|\vec{x}\|_p \geq \|\vec{x}\|_q \geq c\|\vec{x}\|_p.
\]
This follows immediately from D.6. It may be worthwhile to find the best possible constants \( c, C \), but we will not do that here.

**D.2 Norms and metrics**

**Definition D.8.** A real normed linear space will consist of an \( \mathbb{R} \) vector space \( V \) and a norm \( \| \cdot \| : V \to \mathbb{R} \) with the following properties. For all \( v, w \in V \) and \( r \in \mathbb{R} \):

1. \( \|v\| \geq 0 \), with equality if and only if \( v = 0 \).
2. \( \|rv\| = |r|\|v\| \).
3. \( \|v + w\| \leq \|v\| + \|w\| \).

An obvious complex analogue of the above also may be defined.

**Proposition D.9.** For any \( n \in \mathbb{N} \) and any \( p \in [0, \infty) \), the pair \( (\mathbb{R}^n, \| \cdot \|_p) \) defined in the previous section is a normed linear space.

**Proof.** Easy. \( \square \)

**Proposition D.10.** If \( (V, \| \cdot \|) \) is a normed linear space, then the function \( d(v, w) = \|v - w\| \) defines a metric on \( V \).

**Proof.** This is not at all difficult.

1. Property 1 of Definition D.8 implies immediately that \( d(x, y) \geq 0 \) with equality if and only if \( x = y \).
2. Property 2 of Definition D.8 with \( r = -1 \) shows that
\[
d(x, y) = \|x - y\| = \|y - x\| = d(y, x).
\]
3. Property 3 of Definition D.8 applies to give
\[
d(x, y) = \|x - y\| = \|(x - z) - (y - z)\| \leq d(x, z) + d(y, z).
\]
\( \square \)

**Notation D.11.** The notation \( d_p \) is used for the metric associated to the normed linear space \( (\mathbb{R}^n, \| \cdot \|_p) \).


D.3 The \( p \)-norms and product metrics

**Notation D.12.** The notation \((x_n)\) will be used to denote a sequence (finite or infinite) of real numbers indexed by a natural number \(n\). So \((x_n)\) means the same thing as \((x_1, x_2, x_3, \ldots)\), finite or infinite depending on context. Occasionally, we will have a need to write something complicated like the sequence \((m/m/2, m/3, \ldots)\) where there is a parameter. In this case we may write the sequence as \((m/n)_n\), where the external ‘\(n\)’ indicates that \(n\) is the variable indexing the terms of the sequence.

**Definition D.13.** Let \(\{(X_1, d_1), \ldots, (X_n, d_n)\}\) be a finite set of metric spaces. Let \(X = \prod_{i=1}^{n} X_i\), let \(p \in [1, \infty]\). Define a function \(d^p : X \times X \to [0, \infty)\) by \(d^p((x_i), (y_i)) = \|((d_i(x_i, y_i))\|_p\).

**Proposition D.14.** The functions \(d^p\) defined above are all metrics.

*Proof.* Symmetry is immediate. If \(d^p((x_i), (y_i)) = \|(d_i(x_i, y_i))\|_p = 0\), then \(d_i(x_i, y_i) = 0\) for all \(i\), and since \(d_i\) is a metric, this implies \((x_i) = (y_i)\). The triangle inequality is given by combining the triangle inequalities for each \(d_i\) metric with that for \(\|\cdot\|_p\), and noting that \(\|\cdot\|_p\) is increasing in each variable:

\[
d^p((x_i), (y_i)) = \|(d_i(x_i, y_i))\|_p \leq \|(d_i(x_i, z_i)) + (d_i(z_i, y_i))\|_p \leq \|(d_i(x_i, z_i))\|_p + \|(d_i(z_i, y_i))\|_p = d^p((x_i), (z_i)) + d^p((z_i), (y_i))
\]

\(\Box\)

**Proposition D.15.** The metrics \(d^p\) all generate the same topologies.

*Proof.* It suffices to show that for any \(p, p' \in [1, \infty]\), any ball \(B_p((x_i), \epsilon)\) for the \(d^p\) metric with \(\epsilon > 0\) contains a ball \(B_{p'}((x_i), \eta)\) for the \(d^{p'}\) metric with \(\eta > 0\) and having the same centre.

We know from Proposition D.7 that there is some constant \(c > 0\) such that \(c\|((d(x_i, y_i)))\|_p \leq \|(d(x_i, y_i))\|_{p'}\). Then \(B_{p'}((x_i), c\epsilon) \subseteq B_p((x_i), \epsilon)\).

\(\Box\)

**Exercise D.16.** The metric \(d^{\infty}\) generates the product topology; therefore all the metrics \(d^p\) generate the product topology.

**Remark D.17.** It is easily seen that \(d_p\) on \(\mathbb{R}^n\) from Notation D.11 is the product metric \(d^p\) for \(n\) copies of \((\mathbb{R}, |\cdot|)\). By reference to Proposition D.15, the metric spaces \((\mathbb{R}^n, d_p)\) and \((\mathbb{R}^n, d_{p'})\) all generate equivalent topologies for all \(p, p' \in [1, \infty]\), and this topology is the product topology on \(\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}\).
D.4 The $p$-norms on sequence spaces

Definition D.18. If $p \in [1, \infty)$, we define a set $\ell^p \subset \prod_{i=1}^{\infty} \mathbb{R}$ to consist of those sequences $(x_n)$ such that
\[ \sum_{i=1}^{\infty} |x_i|^p \]
converges to a real number. For such a sequence, we define
\[ \| (x_n) \|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}. \]

Proposition D.19. The pair $(\ell^p, \| \cdot \|_p)$ is a normed linear space.

Proof. Conceptually, in the first place we must show that $\ell^p$ is a vector subspace of $\prod_{i=1}^{\infty} \mathbb{R}$. We must show it is closed under addition of vectors and under scalar multiplication. In the second, we must show that $\| \cdot \|_p$ has the properties of a norm. In practice, it is simpler to prove all these facts concerning addition together then all the facts concerning scalar multiplication.

Suppose $(x_n)$ and $(y_n)$ are sequences in $\ell^p$, then for all $N \in \mathbb{N}$
\[ \left( \sum_{i=1}^{N} |x_i + y_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^{N} |x_i|^p \right)^{1/p} + \left( \sum_{i=1}^{N} |y_i|^p \right)^{1/p} \]
by the Minkowski inequality.

Rearranging this, we deduce
\[ \sum_{i=1}^{N} |x_i + y_i|^p \leq \left( \| (x_n) \|_p + \| (y_n) \|_p \right)^{p} \cdot \]
The right hand side is the $N$-th partial sum of the series
\[ \sum_{i=1}^{\infty} |x_i + y_i|^p \] (D.4)
which consists of positive terms. The left hand side is independent of $N$, and therefore we deduce that (D.4) converges, and the limit is less than or equal to $(\| (x_n) \|_p + \| (y_n) \|_p)^{p} \cdot$
Rearranging, we deduce that
\[ \cdot (x_n) + (y_n) \in \ell^p \]
Appendix D. \( p \)-Norms

- \( \| (x_n) + (y_n) \|_p \leq \| (x_n) \|_p + \| (y_n) \|_p \).

As for scalar multiplication, it is straightforward to show that

\[
\| r(x_n) \|_p = \left( \sum_{i=1}^{\infty} |r x_i|^p \right)^{1/p} = |r| \| (x_n) \|_p
\]

which shows that

- \( r(x_n) \in \ell^p \),
- \( \| r(x_n) \|_p = |r| \| (x_n) \|_p \).

Finally, we observe that \( \| (x_n) \| = 0 \) if and only if every term of \( (x_n) \) is 0. \( \square \)

**Definition D.20.** We define \( \ell^\infty \) to consist of those sequences \( (x_n) \) such that \( \sup_{i \in \mathbb{N}} |x_i| < \infty \), i.e. the bounded sequences. We define \( \| (x_n) \|_\infty \) as \( \sup_{i \in \mathbb{N}} |x_i| \).

**Exercise D.21.** With the definitions above \((\ell^\infty, \| \cdot \|_\infty)\) forms a normed linear space.

**Definition D.22.** Let \( c \) denote the set of convergent sequences of real numbers, \( c_0 \) the set of sequences of real numbers with limit 0, and \( \mathbb{R}^\infty \) or \( c_{00} \) the set of sequences of real numbers having at most finitely many nonzero terms.

Unless otherwise specified, we give \( \ell^p \) the topology induced by the (metric induced by the) norm \( \| \cdot \|_p \). We give \( c \) and \( c_0 \) the topologies inherited from \( \ell^\infty \). Which norm, metric or topology one should place on \( \mathbb{R}^\infty \) is less clear, see Exercise D.30.

**Proposition D.23.** Suppose \( p, q \in [1, \infty) \) satisfy \( p < q \). Then there are strict inclusions

\[
\mathbb{R}^\infty \subset \ell^p \subset \ell^q \subset c_0 \subset c \subset \ell^\infty
\]

and if \( (x_n) \in \ell^p \), then \( \| (x_n) \|_p \geq \| (x_n) \|_q \geq \| (x_n) \|_\infty \).

**Proof.** We prove this in several steps:

1. \( \mathbb{R}^\infty \subset \ell^1 \). The inclusion is immediate, and considering the sequence \( (x_n) = (1/2, 1/4, 1/8, \ldots) \) for which \( \| (x_n) \|_1 = 1 \) but which is not in \( \mathbb{R}^\infty \) shows that it is strict.
D.4. The $p$-norms on sequence spaces

2. Suppose $p < q \in [1, \infty)$. Suppose $(x_n) \in \ell^p$. For any initial sequence, we have

$$\sum_{i=1}^{N} |x_i|^q \leq \left( \sum_{i=1}^{N} |x_i|^p \right)^{q/p}$$

since $\bar{x} \mapsto \|\bar{x}\|_p$ for $\bar{x} \in \mathbb{R}^N$ is decreasing as a function of $p$. But this implies that, in the limit,

$$\sum_{i=1}^{\infty} |x_i|^q \leq \|(x_n)\|_p^q,$$

from which the inclusion $\ell^p \subset \ell^q$ and the inequality $(x_n)\|_p \geq \|(x_n)\|_q$ both follow.

We observe that if $x_n = 1/(n)^{1/p}$, then $\sum_{i=1}^{\infty} |x_n|^p = \sum_{i=1}^{\infty} 1/n$ diverges but $\sum_{i=1}^{\infty} |x_n|^q = \sum_{i=1}^{\infty} 1/n^{q/p}$ converges, both by the integral test for convergence. So the inclusion is strict.

3. If $(x_n) \in \ell^q$, then the series $\sum_{i=1}^{\infty} |x_i|^q$ converges, so $\lim_{i \to \infty} x_i = 0$, so $(x_n) \in c_0$.

The sequence $x_n = 1/\log(n+1)$ shows that the inclusion is strict.

4. Any sequence converging to 0 converges, so $c_0 \subset c$. The inclusion is clearly strict, since the constant sequence 1 converges, but not to 0.

5. Any convergent sequence is bounded, so $c \subset \ell^\infty$. The sequence $(-1)^n$ is bounded but not convergent.

6. Finally, we show that if $(x_n) \in \ell^p$ for $p < \infty$, then $|x_i| \to 0$ as $i \to \infty$. Assume $(x_n) \neq 0$, since the case of 0 is trivial. In particular, if $s = \sup_i |x_i| > 0$ is not attained, then there is some subsequence of $(x_i)$ converging to $s > 0$, a contradiction. So there is some $n$ such that $\sup_{i} |x_i| = |x_n|$, and for this value $n$, we have

$$\| (x_n) \|_\infty = \sup_{i} |x_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \leq \| (x_n) \|_p$$

by reference to Proposition D.6.

\[\Box\]

**Corollary D.24.** If $p < q$ are elements of $[1, \infty]$, then the inclusions $\ell^p \subset \ell^q$ are continuous.
Proof. These are metric spaces, and it suffices to give an $\epsilon - \delta$ proof of continuity. Let $\epsilon > 0$ and choose $\delta = \epsilon$. For any $(x_n), (y_n) \in \ell^p$, if

$$
\| (x_n) - (y_n) \|_p < \delta
$$

then

$$
\| (x_n) - (y_n) \|_q < \delta = \epsilon.
$$

So the inclusion is indeed continuous. \hfill \Box

Another useful fact is the following. Each space appearing in Proposition D.23 is a subspace of the space of all sequences: $\mathbb{R}^N$. This space may be equipped with the product topology.

**Proposition D.25.** Let $p \in [1, \infty]$. Then the inclusion $\ell^p \to \mathbb{R}^N$ is continuous.

**Proof.** The product topology on $\mathbb{R}^N$ is the weakest (or coarsest) topology making each projection continuous. In particular, if the functions $\pi_i : \ell^p \to \mathbb{R}$ that take a sequence $(x_n)$ to the $i$-th term $x_i$ are continuous, then the induced map $\ell^p \to \mathbb{R}^N$ is continuous, and it is easy to verify that this map is indeed the inclusion.

So it suffices to show that $\pi_i$ is continuous. Since the source and target are both metric spaces, an $\epsilon - \delta$ argument applies. Suppose $\| (x_n) - (y_n) \|_p < \epsilon$, then in particular $|x_i - y_i| < \epsilon$, which implies $|\pi_i((x_n)) - \pi_i((y_n))| < \epsilon$. So $\pi_i$ is indeed continuous, and the result follows. \hfill \Box

**Proposition D.26.** Let $p \in [0, \infty)$. The subset $\mathbb{R}^\infty$ is dense in $\ell^p$.

**Proof.** Let $(x_n)$ be a sequence in $\ell_p$, and for $m \in \mathbb{N}$ let $(x_{m,n})_n$ denote the sequence for which $x_{m,n} = x_n$ if $m \leq n$ and $x_{m,n} = 0$ otherwise.

We wish to show that $\ell^p = \overline{\mathbb{R}^\infty}$. We show that every element of $\ell^p$ is a limit of a sequence of elements in $\mathbb{R}^\infty$—this is a sequence of sequences.

Let $\epsilon > 0$. Consider the sequence $((x_{m,n})_m$ of elements of $\mathbb{R}^\infty$. Observe that

$$
\| (x_{m,n})_n - (x_n) \|_p^p = \sum_{i=1}^m |x_i - x_i|^p + \sum_{i=m+1}^\infty |x_i|^p = \sum_{i=m+1}^\infty |x_i|^p.
$$

Since the series $\sum_{i=1}^\infty |x_i|^p$ converges to $\| (x_n) \|_p^p$, we can find some $N$ such that

$$
\left| \sum_{i=1}^m |x_i|^p - \| (x_n) \|_p^p \right| < \epsilon^p
$$
whenever \( m > N \), which is equivalent to

\[
\sum_{i=1}^{m} \left| x_i \right|^p - \sum_{i=1}^{\infty} \left| x_i \right|^p < \epsilon^p,
\]

but this is equivalent to saying

\[
\sum_{i=m+1}^{\infty} \left| x_i \right|^p < \epsilon^p
\]

whenever \( m > N \), which is to say that \( \|(x_{m,n})_n - (x_n)\|_p < \epsilon^p \), and taking \( p \)-th roots, we see that

\[
\|(x_{m,n})_n - (x_n)\|_p < \epsilon
\]

whenever \( m > N \). Therefore the sequence \( ((x_{m,n})_m)_n \to (x_n) \) as \( m \to \infty \). \( \square \)

**Exercise D.27.** Let \( p \in [1, \infty) \). Let \( Q^\infty \subset R^\infty \) denote the set of sequences having only rational-number terms and which are eventually 0.

1. Prove \( Q^\infty \) is dense in \( \ell^p \).

2. Give \( c_0 \) the subspace topology inherited from \( \ell^\infty \). Prove \( Q^\infty \) is dense in \( c_0 \).

**Corollary D.28.** Since \( Q^\infty \) is in bijection with the countable union \( \bigcup_{i=1}^{\infty} Q^i \) of countable spaces, it follows that each of the spaces \( \ell^p \) for \( p \in [1, \infty) \) or \( c_0 \) or \( c \) is separable, and since they are metric, they are second countable.

**Exercise D.29.** Prove that \( \ell^\infty \) is not separable (and therefore, not second countable)

The situation for infinite-dimensional spaces is therefore much more complicated than for finite-dimensional spaces. In the finite-dimensional setting, there was only one linear space for each dimension, \( R^n \), and each of the norms \( \| \cdot \|_p \) induced the same topology. In the infinite-dimensional case, the topologies and spaces on which they are defined are all different.

It can be conceptually helpful to view the elements of \( R^\infty \), that is, finite sequences of some undetermined length, as the objects one is most likely to encounter in practical situations; the real world is generally finitist. Then the different spaces \( \ell^p \), \( c_0 \) and \( c \) are different choices of which sequences of elements in \( R^\infty \) one views as convergent.

**Exercise D.30.** Consider the various normed linear spaces \( (R^\infty, \| \cdot \|_p) \) for \( p \in [1, \infty] \). Prove that these are pairwise inequivalent as metric spaces by considering which sequences of elements in \( R^\infty \) are convergent for the various \( d_p \) metrics.
D.4.1 Completeness

Exercise D.31. Prove that the spaces \((\ell^p, \| \cdot \|_p)\) are complete for all \(p \in [1, \infty]\). What can be said about \(R^\infty\), \(c_0\) and \(c\)?

D.5 The \(p\)-norms for functions

This is not a course in measure theory, so we content ourselves with the following inadequate treatment.

Definition D.32. Let \(p \in [1, \infty)\). Suppose \(f : [a, b] \to \mathbb{R}\) is a function defined on an interval \([a, b]\) for which
\[
\int_a^b |f|^p \, dx
\]
is defined (and finite). Then define

\[
\|f\|_p = \left( \int_a^b |f|^p \, dx \right)^{1/p}.
\]

Remark D.33. The integral should really be taken in the sense of Lebesgue, but we will restrict our attention to piecewise continuous functions on closed bounded intervals, which will allow us to use only Riemann integrals, including improper Riemann integrals if necessary.

Exercise D.34. Let \(P[a, b]\) denote the set of piecewise-continuous functions on the closed bounded interval \([a, b]\). For \(f \in P[a, b]\) and \(p \in [1, \infty)\), show that \((P[a, b], \| \cdot \|_p)\) makes \(P[a, b]\) a normed linear space.

Definition D.35. We say that \(C\) is an essential supremum for a piecewise continuous function \(f : [a, b] \to \mathbb{R}\) if the set \(S\) of values \(x\) for which \(f(x) > C\) does not have any interior points—i.e., \(S\) contains no open intervals.

Exercise D.36. For \(f \in P[a, b]\), define \(\|f\|_\infty\) to be

\[
\|f\|_\infty = \inf \{ C : C \text{ is an essential supremum for } |f| \}.
\]

Prove that \((P[a, b], \| \cdot \|_\infty)\) is a normed linear space.

Exercise D.37. Show that the spaces \((P[a, b], \| \cdot \|_p)\) as \(p\) varies over \([1, \infty]\) are all different.
D.5. The $p$-norms for functions
Appendix E

Urysohn’s Lemma and the Tietze Extension Theorem

E.1 Urysohn’s Lemma

Suppose \( X \) is a topological space and suppose that for any two disjoint closed sets \( C_0, C_1 \subseteq X \) we can find a continuous \( f : X \to \mathbb{R} \) such that \( f|_{C_0} \equiv 0 \) and \( f|_{C_1} \equiv 1 \). Then \( f^{-1}(-\infty, 1/2) \) and \( f^{-1}(1/2, \infty) \) give us two open sets that separate \( C_0 \) and \( C_1 \). This implies \( X \) is normal.

**Theorem E.1** (Urysohn’s Lemma). Suppose \( X \) is a normal topological space and that \( C_0 \) and \( C_1 \) are disjoint closed sets in \( X \). Then there exists a continuous function \( f : X \to [0, 1] \) such that \( f(C_0) = 0 \) and \( f(C_1) = 1 \).

**Proof.** Use normality to produce a nested sequence of open sets \( U_d \), one for each dyadic rational \( d = a/2^n \) in \([0, 1] \cap \mathbb{Q} \), such that \( C_0 \subseteq U_0 \) and \( C_1 \subseteq X \setminus \bar{U}_1 \), and such that \( d < d' \) implies \( \bar{U}_d \subseteq U_{d'} \). Then define \( f : X \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
1 & \text{if } x \notin U_1 \\
\inf_{x \in U_d} d & \text{otherwise}
\end{cases}
\]

To prove the function \( f \) is continuous, let \( x \in X \), and fix \( \epsilon > 0 \). Write \( t = f(x) \). We want to find an open neighbourhood \( V \) of \( x \) such that \( f(V) \subseteq (t - \epsilon, t + \epsilon) \). The cases of \( t = 0 \) and \( t = 1 \) are exceptional. If \( t = 0 \), then we can find some \( d < \epsilon \) such that \( x \in U_d \). Then \( U_d = V \) works. If \( t = 1 \), then we can find \( d > 1 - \epsilon \) such that \( x \notin \bar{U}_d \). Then \( X \setminus \bar{U}_d \) works. Therefore assume \( t \in (0, 1) \).
We can find some diadic numbers $d_1 > d_2 > t - \epsilon$ such that $x \notin U_{d_1} \supset U_{d_2}$. We can also find $d_3 < t + \epsilon$ such that $x \in U_{d_3}$. Then consider $V = U_{d_3} \setminus U_{d_2}$. This is an open set containing $x$, and $f(V) \subseteq [d_2, d_3] \subseteq (t - \epsilon, t + \epsilon)$.

So $f$ is continuous. \hfill \qed

Example E.2. Let $X$ be an uncountable space and let $p \in X$ be a point. Define the fortissimo topology on $X$ as follows: a subset $C \subseteq X$ is closed if $p \in C$ or if $C$ is countable.

We claim this space is normal. Two disjoint closed sets consist of two disjoint countable subsets neither containing $p$, or one countable subset and one subset containing $p$. In the first case, the sets are also open since their complements contain $p$. In the second, the countable set not containing $p$ is open, and its complement is also open.

Therefore Urysohn’s lemma applies to $X$. On the other hand, consider a continuous function $f : X \to \mathbb{R}$ such that $f(p) = 0$. Note that $\{p\}$ itself is a closed point. The sets $U_n = f^{-1}((-1/n, 1/n))$ form a countable nested family of open sets in $X$, each containing $p$. Therefore each $U_n$ is cocountable. It follows that $f^{-1}(0) = \bigcap_n U_n$ is also cocountable, so that in particular, the closed set $\{p\}$ cannot be expressed $f^{-1}(0)$ for any continuous $f : X \to \mathbb{R}$.

Definition E.3. If $X$ is a topological space, a $G_\delta$-set is any subset of $X$ that can be written as an intersection of countably many open subsets.

Definition E.4. A $G_\delta$-space is a topological space $X$ in which every closed subset is a $G_\delta$-set. A space $X$ is perfectly normal if it is normal and a $G_\delta$-space.

Exercise E.5. A space $X$ is perfectly normal if and only if every closed set $C$ is the zero set of a continuous function $f : X \to \mathbb{R}$.

E.2 Tietze Extension

For perfectly normal Hausdorff spaces, this was proved in a sequence of exercises.

Theorem E.6 (Tietze Extension). Let $X$ be a normal topological space, let $C \subseteq X$ be a closed subspace and let $f : C \to [-1, 1]$ be a continuous function. Then there exists a continuous function $F : X \to [-1, 1]$ such that $F(c) = f(c)$ for all $c \in C$.

Remark E.7. Of course, the role of $[-1, 1]$ can be played by any nontrivial closed interval.

Proof. The idea is to produce a sequence of continuous functions $f_n : X \to [-1, 1]$ approximating $f$ on $C$. 
**Lemma E.8.** For any $r > 0$, and any continuous function $h : C \to [-r, r]$, there exists a continuous extension $g : X \to [-r/3, r/3]$ such that

$$|g(a) - h(a)| \leq 2r/3$$

*Proof.* Divide the interval $[-r, r]$ into three equal thirds by closed subintervals: $I_1 \cup I_2 \cup I_3$. Let $C_1 = h^{-1}(I_1)$ and $C_2 = h^{-1}(I_3)$. By Urysohn’s lemma, we can make a continuous function $g : X \to [-r/3, r/3]$ that takes the value $-r/3$ on $C_1$ and $r/3$ on $C_2$. This $g$ has the required property. \qed

Now, the original $f$ satisfies the hypotheses of the lemma with $r = 1$. Define $g_1 : X \to [-1/3, 1/3]$ from the lemma. Set $s_1 = g_1$. Now apply the lemma to $f - s_1$, now with $r = 2/3$, to get $g_2$. Apply the lemma to $f - s_2$ with $r = 4/9$ to get $g_3$ and thus $s_3$ and so on.

Now we have produced an infinite sequence of functions $g_n : X \to [-2^{n-1}/3^n, 2^{n-1}/3^n]$. By an easy calculus argument (comparison with a geometric series), the sequence

$$F(x) = \sum_{i=1}^{\infty} g_i(x) = \lim_{n \to \infty} s_n(x)$$

converges for all $x \in X$.

A little bit of estimation shows that if $n > m$, then $|s_n(x) - s_m(x)| < (2/3)^m$. In particular, $(s_n(x)) \to F(x)$ uniformly, which is sufficient to prove that $F(x)$ is continuous. For any $c \in C$, we know that

$$|f(c) - s_n(c)| \leq (2/3)^n$$

so that $f(c) = \lim_{n \to \infty} s_n(c) = F(c)$.

The same estimation as before shows that $F$ takes values in $[-1, 1]$. \qed

**Lemma E.9.** Let $X$ be a normal topological space and let $C$ be a closed subspace. Suppose $f : C \to \mathbb{R}$ is a continuous function, then there exists a continuous function $F : X \to \mathbb{R}$ such that $F(c) = f(c)$ for all $c \in C$.

*Proof.* Take a homeomorphism $h : \mathbb{R} \to (-1, 1)$. By composing with $h$, we can find $F' : X \to [-1, 1]$ such that $F'|_C = h \circ f$. Let $J$ denote the set of points $x \in X$ such that $|F'(x)| = 1$—these points prevent us from applying $h^{-1}$, so we’ll get rid of them. The set $J$ is closed and disjoint from $C$. By Urysohn’s lemma, we can find a continuous $\chi : X \to [0, 1]$ that takes the value 0 on $J$ and 1 on $C$. The function $x \mapsto \chi(x)F'(x)$ is continuous, and takes values in $(-1, 1)$. Consider $F = h^{-1} \circ (\chi F')$. This has all the properties we asked for. \qed
Index

$G_δ$-set, 126
$G_δ$-space, 126
$T_3$-space, 9
$T_4$-space, 10
$k$-closed, 38
$n$-th homotopy group of $(X, x_0)$, 48
(covariant) functor, 102
(open) neighbourhood, 8
étale map, 71
amalgamated product, 61

base, 12
based topological spaces, 47
bimorphism, 101
boundary of $A$, 20
box, 16
Cauchy, 24
closed, 7
closed embedding, 14
closure, 19
coarser, 14
cocomplete, 109
cofinite topology, 9
coinduced, 16
colimit, 107, 108
commutative, 107
compact, 27
compact–open, 95
compactification, 34
compactly generated, 38
complete, 24, 109
completion, 26
connected, 41, 56
connected component, 42
constant $I$-shaped diagram with value $X$, 107
continuous, 10
contractible, 50
contravariant functor, 102
converges, 22
coproduct, 109
counit, 106
countable, 12
covering space action, 77
covering space map, 71
covering spaces of $X$, 75
curry, 96
deck transformation, 87
deformation retract, 47
dense, 21
diagonal, 21
diagram, 107
diameter, 30
disconnected, 41
discrete group, 91
discrete topology, 9

embedding, 14
epimorphism, 101
equivalence of categories, 83, 104
equivalent, 53
essential, 50
essentially surjective, 103
faithful, 92, 103
fibre, 72
fibre product, 109
final, 37
finer, 14
first countable, 12
fortissimo, 126
free, 78, 92
free product, 61
full, 100, 103
fundamental group, 49, 56
fundamental groupoid of $X$ with endpoints in $A$, 55
generated, 13
group, 55
groupoid, 55
Hausdorff, 9
homeomorphism, 11, 101
homeomorphism of compactifications, 37
homotopic, 45
homotopic relative to $A$, 45
homotopy, 45
homotopy category, 46
homotopy equivalence, 46
homotopy relative to $A$, 45
identity morphism, 99
indiscrete topology, 9
induced, 14
initial, 110
initial object, 52
interior, 20
isometry, 25
isomorphism, 101
large, 99
left $G$-action, 92
left adjoint, 105
left cancellable, 101
lift, 72
limit, 107
local base, 12
locally compact, 35
locally connected, 42
locally path connected, 43
map of compactifications of $X$, 37
map of covering spaces, 82
metric topology, 8
monodromy action, 75
monomorphism, 101
morphism of groupoids, 56
natural transformation, 104
natural transformation, 104
normal, 9, 88
normal covering space, 78
normalizer, 87
nowhere dense, 21
nullhomotopic, 50
one-point compactification, 34
open, 11
open ball around $x$ of radius $r$, 8
open cover, 27
open embedding, 14
open sets, 7
opposite category, 103
pair, 57
path component, 43
path-connected, 43
perfectly normal, 126
pointed contractibility, 50
pointed homotopy category, 48
pointed spaces, 100
product, 109
product category, 101
product space, 15
product topology, 14
proper, 80, 93
pseudometric space, 25
pullback, 110
pushout, 110
quotient map, 89, 93
quotient space, 16, 93
quotient topology, 16
reduced word, 61
regular, 9
retract, 46
retraction, 46
reverse, 54
right $G$-action, 92
right adjoint, 105
right cancellable, 101
right half-open interval topology, 13
saturate, 89
saturated, 81
saturated for $f$, 89
second countable, 12
semilocally simply connected, 86
sequentially closed, 23
sequentially compact, 31
simply connected, 58
small, 100
small skeleton, 105
smash product, 49
source, 99
sparse, 21
split epimorphism, 101
split monomorphism, 101
stabilizer, 92
starshaped, 51
Stone–Čech compactification, 38
strong deformation retract, 47
subbase, 13
subcategory, 100
subspace topology, 14
system of neighbourhoods of $x$, 12
target, 99
terminal, 110
terminal object, 52
the fundamental groupoid, 55
topological group, 91
topological space, 7
topology, 7
topology of uniform convergence on compact subsets, 95
totally bounded, 31
totally disconnected, 42
transitive, 92
trivializes, 72
two-out-of-six, 102
two-out-of-three, 101
uncurry, 97
uniform metric, 96
unit, 106
universal cover, 86
universal covering space, 80
weak Hausdorff, 35
wedge sum, 49
word, 61
zero object, 52, 110
Bibliography