Instructions:

• This is a final exam. You must do it without help from anyone else.

• You may refer to the following sources:
  
  – The class lecture notes (either handwritten or typeset).
  
  – *Topology* by Munkres (all such references must be cited).
  
  – Chapter 0 and 1 of *Algebraic Topology* by Hatcher (all such references must be cited).
  
  – The homework assignments for the course—including the official solutions.

• You may not refer to any other sources.

• If you do not understand a question or think that there may be a mistake in a question, let me know as soon as possible.

• In some cases, proof-by-picture may be sufficient. Please ensure I can understand your picture.

• Marks may be taken off for lack of clarity.

• If a topology on a subset of \( \mathbb{R}^n \) is not otherwise specified, the subspace topology is meant in all cases.

• Questions are in multiple parts. In answering any part of a question, you may assume the results of the preceding parts of that question, even if you have not been able to prove them.

• There are 4 questions, with point breakdowns as indicated. Answer all questions.

1. **6pts.** Let \( N \) denote the natural numbers \( \{1, 2, 3, \ldots \} \) and define 

\[
U_a(b) = \{ b + na \mid n \in \mathbb{Z} \} \cap N.
\]

(a) Consider the sets \( U_a(b) \) where \( a, b \) range over relatively prime natural numbers. Show that these sets form a base for a topology \( \tau \) on \( N \). Call the resulting topological space \( X \).

(b) Show that \( X \) is Hausdorff.

(c) Determine, with proof, the set of continuous maps \( f : X \to \mathbb{R} \), where \( \mathbb{R} \) is given the usual topology.

2. **8pts.** Suppose \( X \) is a non-compact but compactly generated Hausdorff space. In particular, a subset \( A \subset X \) is closed if and only if \( A \cap K \) is closed in \( K \) for all compact subsets \( K \) of \( X \) (in the presence of the Hausdorff property, this is equivalent to the definition of compactly generated given in the lecture notes. You do not need to prove this).
(a) Prove that the one-point compactification \( X \cup \{\infty\} \) is weakly Hausdorff, which is to say that the image of any continuous function \( f : Y \to X \cup \{\infty\} \) having compact Hausdorff domain is closed. Hint: show any compact subset of \( X \cup \{\infty\} \) is closed.

(b) Prove that if \( Z \) is a weakly Hausdorff space and if \((z_n)\) is a sequence in \( Z \), then \((z_n)\) has at most one limit.

(c) Give an example with proof of a weakly Hausdorff space that is not Hausdorff.

3. 8pts. The (closed) Möbius band \( M \) is a topological space obtained as a quotient of \( I \times I \) by the relation \((0, y) \sim (1, 1-y)\) for all \( y \in I \) (see Figure 1). It is a compact surface with boundary, and is often seen embedded in \( \mathbb{R}^3 \) as in Figure 2.

![Figure 1: The identifications required to produce \( M \) as a quotient of \( I \times I \).](image1.png)

![Figure 2: A standard embedding of the Möbius band in \( \mathbb{R}^3 \).](image2.png)

(a) Prove that \( M \) is not homeomorphic to the closed unit disk, \( D^2 \subset \mathbb{R}^2 \).

(b) Prove that \( M \) is not homeomorphic to the closed cylinder, \( S^1 \times I \).

(c) By contrast with the case of the (singly-twisted) Möbius band \( M \), the doubly-twisted Möbius band is the image of an unusual closed embedding \( d : S^1 \times I \to \mathbb{R}^3 \), obtained by twisting a ribbon twice and joining the ends together to form a loop with two twists. The image of such an embedding is depicted in Figure 3 and a photograph of a doubly-twisted Möbius band is given in Figure 5. Write \( Y \) for the subspace that is the image of \( d \) in \( \mathbb{R}^3 \). Let \( X \) be a cylinder embedded in the usual way in \( \mathbb{R}^3 \), as depicted in Figure 4. Prove that there is no ambient isotopy of \( \mathbb{R}^3 \) taking \( Y \) to \( X \).

4. 8pts. Let \( V \) denote an \( \mathbb{R} \)-vector space consisting of infinite matrices

\[
\begin{bmatrix}
a_1 & a_2 & a_3 & \ldots \\
b_1 & b_2 & b_3 & \ldots
\end{bmatrix}
\]

where all except finitely many of the entries are 0. A metric \( d \) can be defined on \( V \) so that \( d(A, A') \) is the supremum of the distances between corresponding entries of \( A \) and \( A' \). You may assume this is a
metric and that the vector space operations of addition and scalar multiplication are continuous when \( V \) is given this metric.

Let \( U \) denote the subspace of \( V \) consisting of matrices \( A \) such that the columns of \( A \) span \( \mathbb{R}^2 \).

Let \( S : V \to V \) be the double-shift linear transformation

\[
S \left( \begin{bmatrix} a_1 & a_2 & a_3 & \ldots \\ b_1 & b_2 & b_3 & \ldots \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & a_1 & a_2 & a_3 & \ldots \\ 0 & 0 & b_1 & b_2 & b_3 & \ldots \end{bmatrix}
\]

An \( \epsilon-\delta \) argument shows that \( S \) is continuous, and you may assume this. Note that \( S \) restricts to give a self map \( S : U \to U \) (denoted by the same letter in an abuse of notation).

(a) Prove that \((1 - t)\text{id}_U + tS\) defines a homotopy from \(\text{id}_U\) to \(S : U \to U\).

(b) Prove that \(U\) is contractible.

(c) Suppose \( G \) is a finite subgroup of \( GL_2(\mathbb{R}) \), i.e., \( G \) is a finite group consisting of invertible \( 2 \times 2 \) matrices, or equivalently of linear symmetries of \( \mathbb{R}^2 \). Using \( U \), produce, with proof, a Hausdorff topological space having fundamental group isomorphic to \( G \).
Figure 5: A photograph of a doubly-twisted Möbius band