1. Let $X$ be a connected, locally path connected and semilocally simply connected space. Let $f : \tilde{X} \to X$ be a universal cover of $X$, and let $\xi \in \tilde{X}$ be a basepoint, having image $x_0 \in X$. The group $\pi_1(X, x_0)$ acts on $\tilde{X}$ via deck transformations

(a) Prove that the deck transformation action is a covering space action in the sense of Homework 6, and that $f : \tilde{X} \to X$ is the quotient map for this group action.

(b) Suppose $K \subseteq \pi_1(X, x_0)$ is a subgroup. Show that in the induced factorization of $f$

$$\tilde{X} \to \tilde{X}/K \to X$$

the map $\tilde{X}/K \to X$ is a covering space map.

(c) Describe the monodromy action of $\pi_1(X, x_0)$ on the fibre $F_{x_0}(\tilde{X}/K)$.

2. Let $n \geq 1$. Define a space $V$ to be a subspace of $\prod_{i=1}^{\infty} \mathbb{R}^n$ consisting of sequences $(v_1, v_2, \ldots, v_k, 0, \ldots)$ where

(i) Almost all the $v_i$ are 0.

(ii) The span of the set $\{v_1, v_2, \ldots\}$ is $\mathbb{R}^n$.

Consider the shift map $\sigma : V \to V$ defined by $\sigma(v_1, v_2, \ldots) = (0, v_1, v_2, \ldots)$.

(a) Prove that $\sigma$ is homotopic to the identity map.

(b) Prove that $\sigma^n$ is homotopic to the constant map sending all elements to $v_0 = (e_1, e_2, \ldots, e_n, 0, 0, \ldots)$. Deduce that $V$ is contractible.

(c) Recall that a faithful left action of a group $G$ on a set $S$ is an action such that if $gs = s$ for all $s \in S$, then $g = e$. Let $G$ be a finite group acting faithfully on $\mathbb{R}^n$ on the left. There is an induced diagonal action of $G$ on $V$ given by $g(v_1, v_2, \ldots) = (g v_1, g v_2, \ldots)$. Prove this is a covering space action. What is the fundamental group of the quotient? Any basepoint will do.

3. Answers to some parts of this question may be helpful for subsequent questions.

(a) Suppose $U, V$ form an open cover of a space $X$ such that $U$, $V$ are each simply connected and that $U \cap V$ is a disjoint union of two nonempty sets $A_1$ and $A_2$, each simply connected. Let $a_1 \in A_1$ and $a_2 \in A_2$. By using the van Kampen Theorem for groupoids and the set of basepoints $\{a_1, a_2\}$, prove that $\pi_1(X, a_1) \cong \mathbb{Z}$.

(b) Consider the two circles $S_- = \{(x, y, -1) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $S_+ = \{(x, y, 1) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$. Choose a basepoint for $\mathbb{R}^3 \setminus (S_- \cup S_+)$, e.g. $x_0 = (0, 0, 0)$. Calculate $\pi_1(\mathbb{R}^3 \setminus (S_- \cup S_+), x_0)$. 


(c) The space \( \mathbb{R}^3 \setminus (S_- \cup S_+) \) has inclusions \( i_- : \mathbb{R}^3 \setminus (S_- \cup S_+) \to \mathbb{R}^3 \setminus S_- \) and \( i_+ : \mathbb{R}^3 \setminus (S_- \cup S_+) \to \mathbb{R}^3 \setminus S_+ \). Construct a class in \( \pi_1(\mathbb{R}^3 \setminus (S_- \cup S_+), x_0) \) that is not trivial, but has trivial image under \( i_-^* \) and \( i_+^* \).

4. This question shows you how to calculate fundamental groups of surfaces:

(a) The Klein bottle \( K \) is the surface obtained by identifying opposite sides of \( I \times I \) using the two identifications \((0, t) \sim (1, 1 - t)\) and \((s, 0) \sim (s, 1)\), see Figure 1. This is famously a non-orientable surface that cannot be embedded (but can be immersed) in \( \mathbb{R}^3 \), see Figure 2. The Klein bottle can be covered by two open sets, \( U \) (indicated in pale blue) and \( V \) (indicated in pale green). Observe that \( V \) is contractible, \( U \) has \( S^1 \vee S^1 \) as a deformation retract and \( U \cap V \simeq S^1 \).

Consider Figure 3. The indicated loop, \( \rho \), represents a generator \( [\rho] \in \pi_1(U \cap V, x_0) \). Write \( R \) for the image of \([\gamma]\) under inclusion in \( \pi_1(U, x_0) \), the free group on \( \alpha, \beta \). Prove that \( \pi_1(K, x_0) \) is the quotient group generated by \( \alpha, \beta \) subject to relation \( R = e_{x_0} \).

(b) The loop \( \rho \) is homotopic in \( U \) to a composite of four loops as indicated in Figure 4. In order: the red, the purple, the green, the orange. Use this factorization to determine the image of \( R \) in \( \pi_1(U, x_0) \) explicitly in terms of \( \alpha, \beta \), and hence give an explicit presentation of \( \pi_1(K, x_0) \).

(c) Calculate the fundamental group of the genus-2 orientable surface, as in Figure 5. Possible hint: the surface obtained by removing a small disk from the torus \( S^1 \times S^1 \) is homotopy-equivalent to \( S^1 \vee S^1 \).
5. Fix a basepoint $s_0$ for the torus $S^1 \times S^1$. Consider the torus embedded in the standard way in $\mathbb{R}^3$, denoted $i : S^1 \times S^1 \to \mathbb{R}^3$, denoted in Figure 6. The torus has universal cover $\mathbb{R} \times \mathbb{R}$, and the map $f : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ can be arranged so that $f^{-1}(s_0) = \mathbb{Z} \times \mathbb{Z}$. This is also the fundamental group of the torus, based at $s_0$. Let $p, q$ be relatively prime positive integers, and consider the (closed) line segment $\kappa : I \to \mathbb{R} \times \mathbb{R}$ starting at $(0, 0)$ and ending at $(p, q)$. The image of $\kappa$ under $f$ in $S^1 \times S^1$ is a loop based at $s_0$, and the image of $\kappa$ in $\mathbb{R}^3$ is called a $(p, q)$-torus knot. It will be denoted $K_{p, q}$.

(a) Choose a basepoint $t_0 \in S^1 \times S^1 \setminus \text{im}(\kappa)$. Calculate $\pi_1(S^1 \times S^1 \setminus \text{im}(\kappa)), t_0)$. Figure 7 may be helpful here.

(b) Thicken the torus in $\mathbb{R}^3$ slightly by taking an open $\epsilon$-ball around each point for some arbitrarily small $\epsilon > 0$. Thicken the knot by taking a closed $\epsilon$-ball around each point, and denote the thickened knot by $K_{p, q}^\epsilon$. We decompose $\mathbb{R}^3 \setminus K_{p, q}^\epsilon$ into three parts. An open set $T$ consisting of the complement of $K_{p, q}^\epsilon$ in the (thickened) torus, a set $N$, consisting of the volume contained within the thickened torus, and $U$ the unbounded region outside the thickened torus. Write down the fundamental groups of both of the following open sets of $\mathbb{R}^3$, all based at $t_0$:

i. $\pi_1(N \cup T, t_0)$;
ii. $\pi_1(U \cup T, t_0)$;

(c) The complement $S^1 \times S^1 \setminus \text{im}(\kappa)$ is homotopy equivalent to $T$ (the ‘thickened’ complement). Describe the maps $\pi_1(T, t_0) \to \pi_1(N \cup T, t_0)$ and $\pi_1(U \cup T, t_0)$ induced by the inclusions.

(d) Calculate the homotopy group $G_{p, q} = \pi_1(\mathbb{R}^3 \setminus K_{p, q}, t)$.  

Figure 2: An immersion of the Klein bottle in $\mathbb{R}^3$. 

3
(e) Prove that when $p = 2$ and $q = 3$, that $G_{p,q}$ is not abelian. Deduce that $\mathbb{R}^3 \setminus K_{p,q}$ is not homotopy equivalent to the complement of an unknotted circle in $\mathbb{R}^3$. The group $G_{2,3}$ is the knot group of the knot $K_{2,3}$, a trefoil knot.
Figure 4: A factorization, up to homotopy, of $\rho$ in $U$

Figure 5: The genus-2 orientable surface

Figure 6: The standard embedding of the torus in $\mathbb{R}^3$. A torus knot is drawn in red.
Figure 7: A diagram of the complement of a torus knot (red) in the surface of the torus. A deformation retract is suggested in green.