1. This question requires the definition of $\mathbb{C}P^n$, as given in homework 5. It may be helpful to refer to that homework and what was proved there.

Given a topological space $X$ and a natural number $n$, define a space $\text{Sym}^n X$ as the quotient space of $X^n$, the $n$-fold product of $X$ with itself, by the relation

$$(x_1, \ldots, x_n) \sim (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$$

where $\pi$ is any permutation—i.e., any rearrangement—of $\{1, \ldots, n\}$.

(a) Given two topological spaces $X$, $Y$, state which continuous functions $f : X^n \to Y$ factor through the quotient map $q : X^n \to \text{Sym}^n X$, that is, for which maps $f$ can $f$ be written as $f \circ q$ for some continuous $f : \text{Sym}^n X \to Y$.

(b) Let ‘$X$’ denote a formal variable. Write elements of $\mathbb{C}P^{n+1}$ as $a_0 + a_1 X + \cdots + a_n X^n$ as a synonym for $(a_0, a_1, \ldots, a_n)$. You may assume that the polynomial multiplication map $m : (\mathbb{C}^2)^n \to \mathbb{C}^{n+1}$

given by

$$m((a_{1,0} + a_{1,1} X), (a_{2,0} + a_{2,1} X), \ldots, (a_{n,0} + a_{n,1} X)) \mapsto \prod_{i=1}^{n} (a_{i,0} + a_{i,1} X)$$

is continuous. Prove that $m$ induces a continuous map $\tilde{m} : \text{Sym}^n (\mathbb{C}^2 \setminus \{0\}) \to (\mathbb{C}^{n+1} \setminus \{0\})$.

(c) Prove that $\tilde{m}$ induces a continuous bijection $\tilde{m} : \mathbb{C}P^n \to \mathbb{C}P^n$.

(d) Prove that $\tilde{m}$ is a homeomorphism. What does the continuity of $\tilde{m}^{-1}$ imply about the roots of a complex polynomial?

(a) By the universal property of quotient spaces, $f$ can be written as a composite $\tilde{f} \circ q$ if and only if $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for all permutations $\pi$ of $\{1, \ldots, n\}$. That is, $f(x_1, \ldots, x_n)$ does not depend on the order of $x_1, \ldots, x_n$.

(b) This is immediate from the previous part and the observation that the product

$$(a_{1,0} + a_{1,1} X)(a_{2,0} + a_{2,1} X) \cdots (a_{n,0} + a_{n,1} X)$$

does not depend on the order of the factors.

(c) Recall that the space $\mathbb{C}P^n$ is the space of equivalence classes of $n + 1$-tuples $(a_0, \ldots, a_n)$ by the relation $\lambda(a_0, \ldots, a_n) \sim (a_0, \ldots, a_n)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.
Let \( r_n : C^{n+1} \setminus \{0\} \to \mathbb{C}^n \) denote the quotient map. There exists a continuous map \( \tilde{m} \) is immediate from the observation that the class of

\[
r_n(\tilde{m}((a_{1,0} + a_{1,1}X)(a_{2,0} + a_{2,1}X) \cdots (a_{n,0} + a_{n,1}X)))
\]

(i.e., the polynomial considered up to multiplication by nonzero complex scalars) is not changed if we replace \((a_{i,0} + a_{i,1}X)\) by \(\lambda(a_{i,0} + a_{i,1}X)\) where \(\lambda\) is a nonzero complex scalar. Now we knot the map \( \tilde{m} : \text{Sym}^n \mathbb{C}P^1 \to \mathbb{C}^n \) is continuous, by the previous part. We claim it is bijective. The Fundamental Theorem of Algebra states that a nonzero complex polynomial of degree \(\leq n\) has a factorization into \(n\) nonzero terms of the form \(a + bX\), and since \(\mathbb{C}[X]\) is a unique factorization domain, this factorization is unique up to the reordering of factors and multiplication by units in \(\mathbb{C}[X]\), i.e., by \(\mathbb{C} \setminus \{0\}\). This is equivalent to the statement that \(\tilde{m}\) is a bijection.

(d) Since \(\text{Sym}^n \mathbb{C}P^1\) is the image of a compact space \((\mathbb{C}P^1)^{\times n}\) and since \(\mathbb{C}^n\) is Hausdorff (by homework 5), it follows this continuous bijection is a homeomorphism.

Consider a complex polynomial of degree \(n\). This polynomial yields an equivalence class in \(\mathbb{C}P^n\), indeed, in the open set \(U_n \cong \mathbb{C}^n\) where \(a_n \neq 0\). Under the homeomorphism \(\tilde{m}\), this corresponds to a factorization where none of the factors is of the form \((a + 0X)\), i.e., each linear factor lies in \(\mathbb{C} \cong U_1 \subset \mathbb{C}P^1\). The map \(\tilde{m}\) restricts to a homeomorphism \(\text{Sym}^n \mathbb{C}^1 \to \mathbb{C}^n\) when we consider only polynomials of degree exactly \(n\) (rather than polynomials of degree \(n\) or less). Each linear factor corresponds to a root in the complex plane. The continuity of the inverse of \(\tilde{m}\) implies that the unordered \(n\)-tuple of roots of a complex polynomial \(f\) of degree \(n\) depends continuously on the coefficients of \(f\).

\[\square\]

2. Given a closed interval \(A = [a, b]\), the open middle third will denote the interval \(m(A) = (a + (b - a)/3, a + 2(b - a)/3)\). Define closed subsets \(C_n\) of the real line as follows:

- \(C_0 = [0, 1]\)
- \(C_{n-1}\) consists of a disjoint union of finitely many compact intervals, \(C_{n-1} = I_1 \cup \cdots \cup I_N\). Define \(C_n\) to be the union \((I_1 \setminus m(I_1)) \cup (I_2 \setminus m(I_2)) \cup \cdots \cup (I_N \setminus m(I_N))\), that is, the result of discarding the open middle thirds of all the closed intervals appearing in the presentation of \(C_{n-1}\).

Define the Cantor set to be \(C = C_\infty = \bigcap_{n=0}^{\infty} C_n\).

There is an alternative description of \(C_n\) and of \(C\) as follows. It is possible to write each element of \([0, 1]\) in base-3 notation

\[
x = [0.a_1 a_2 a_3 \ldots]_3
\]

where \(a_i \in \{0, 1, 2\}\), and

\[
[0.a_1 a_2 a_3 \ldots]_3 = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \ldots
\]

Just as with decimal notation, some elements have two representations: \([0.0222 \ldots]_3 = [0.1]_3\). Then \(C_1\) consists of those elements having at least one ternary representation starting \([0.0\ldots]_3\) or \([0.2\ldots]_3\),
i.e., not having 1 in the first place. The set $C_2$ is the set of numbers having (at least one) ternary representation not having a 1 in the first or second place, and so on for all $C_n$. The set $C$ is the set of $x \in [0,1]$ having a ternary representation with no 1s. You do not have to prove this is an equivalent definition.

(a) Prove $C$ is a compact, uncountable and nowhere dense subset of $\mathbb{R}$.

(b) Let $X$ denote the space $\prod_{n=1}^{\infty} \{0,2\}$, where $\{0,2\}$ has the discrete topology. There is an evident function $f : X \to C$ given by $f(a_1,a_2,a_3,\ldots) = [0.a_1a_2a_3\ldots]_3$. Prove that $f$ is a homeomorphism.

(a) Each $C_n$ is a closed set, so that $\bigcap_n C_n$ is closed. It is clearly bounded, so $C$ is compact.

For uncountability, imagine for the sake of contradiction that one had an enumeration $\{c_1,c_2,\ldots\}$ of the elements of $C$. Form an element $d$ as follows: if the $n$-th ternary digit of $c_n$ is 0, then the $n$-th ternary digit of $d$ is 2, and vice versa. Then $d$ has no 1s in its ternary representation, so lies in $C$. On the other hand, $d \neq c_i$ for all $i$, a contradiction.

A set is nowhere dense if its closure has empty interior. Since $C$ is closed, it suffices to show it has empty interior. Let $c_1$ and $c_2$ be two elements of $C$ with $c_1 < c_2$. Let $n$ be the first place where the ternary digits of $c_1$ and $c_2$ (both written without 2s). Then choose the number $q$ having the same ternary representation as $c_1$ up until and including the $n-1$st position, then having 1s thereafter. Immediately $q \notin C$, and $c_1 < q < c_2$. Therefore there can be no open intervals contained in $C$, and $C$ is consequently nowhere dense.

(b) The evident function is bijective. We prove that $f^{-1}$ is continuous. It suffices to show that $\pi_n \circ f^{-1}$ is continuous, where $n$ is projection onto the $n$-th factor. That is, it suffices to prove the function $C \to \{0,2\}$ given by reading off the $n$-th ternary digit is continuous. Equivalently, given an element $c \in C$ having a 0 (respectively, a 2) in the $n$-th place, there is an open interval $I$ around $c$ such that $I \cap C$ consists of numbers having a 0 (resp. 2) in the $n$-th place, but this is elementary.

Therefore $f^{-1}$ is a continuous bijection where the source is compact and the target is Hausdorff, so it is a homeomorphism.

3. Let $X$, $Y$ and $Z$ be topological spaces, and assume $X$ is locally compact and Hausdorff (LCH). All spaces of continuous functions are given the compact–open topology.

(a) Consider the maps $\mathcal{C}(X,Y \times Z) \to \mathcal{C}(X,Y)$ and $\mathcal{C}(X,Y \times Z) \to \mathcal{C}(X,Z)$ induced by the projection maps. Prove that the induced map $\mathcal{C}(X,Y \times Z) \to \mathcal{C}(X,Y) \times \mathcal{C}(X,Z)$ is a homeomorphism.

(b) Suppose $(Y,y_0)$ is a pointed space. Let $s \in S^1$ be a basepoint for $S^1$. Write $\mathcal{C}_s(S^1,Y)$ for the set of continuous $f : S^1 \to Y$ such $f(s) = y_0$, given the subspace topology as a subspace of $\mathcal{C}(S^1,Y)$. Recall that $\pi_0(X)$ denotes the set of path components of $X$. What familiar object is $\pi_0(\mathcal{C}_s(S^1,Y))$?
(c) Prove that if \((Y, y_0), (Z, z_0)\) are pointed spaces, then \(\pi_1(Y \times Z, (y_0, z_0)) \cong \pi_1(Y, y_0) \times \pi_1(Z, z_0)\).

(a) First of all, there are continuous projection maps \(p^1 : Y \times Z \to Y\) and \(p^2 : Y \times Z \to Z\). These induce continuous maps \(p^1 : \mathcal{C}(X, Y \times Z) \to \mathcal{C}(X, Y)\) and \(p^2 : \mathcal{C}(X, Y \times Z) \to \mathcal{C}(X, Z)\), by the formulat \(p^1(f)(x) = p^1(f(x))\) and similarly for \(p^2\). Therefore, by the universal property of the product, there is a continuous function \(s : \mathcal{C}(X, Y \times Z) \to \mathcal{C}(X, Y) \times \mathcal{C}(X, Z)\).

We claim \(s\) is a bijection. It is easy to show it is injective; if \(f \neq g\), then there exists some \(x \in X\) such that \(f(x) \neq g(x)\), which implies at least one of \(p^1(f(x)) \neq p^1(g(x))\) or \(p^2(f(x)) \neq p^2(g(x))\), so that \(s(f) \neq s(g)\).

To show it is surjective, suppose \(f : X \to Y\) and \(g : X \to Z\) are continuous functions, then the universal property of the product implies there exists a unique continuous function \(h : X \to Y \times Z\) such that \(f\) and \(g\) are obtained by composing \(h\) with the projections. So \(s\) is surjective.

Therefore we can define an inverse function \(c\) for \(s\). Namely \(c(f, g)\) is the continuous function \(c(f, g)(x) = (f(x), g(x))\). We claim \(c : \mathcal{C}(X, Y) \times \mathcal{C}(X, Z) \to \mathcal{C}(X, Y \times Z)\) is actually continuous in the compact–open topology.

Let \(K\) be a compact subset of \(X\) and \(U\) an open subset of \(Y \times Z\). Let \(S = S(K, U)\) denote the sub-basic open set of \(\mathcal{C}(X, Y \times Z)\) consisting of functions such that \(h(K) \subseteq U\). Let \((f, g) \in c^{-1}(S)\). Then for each \(x \in K\), there exist open sets \(M_x \subseteq Y\) and \(N_x \subseteq Z\) such that \((f, g)(x) \in M_x \times N_x \subseteq U\), since open sets of this form constitute a basis for the product topology on \(Y \times Z\). For each \(x \in K\), choose \(M_x, N_x\), and choose a compact neighbourhood \(J_x \subseteq f^{-1}(M_x) \cap g^{-1}(N_x)\)---i.e., a compact set \(J_x\) containing an open neighbourhood of \(x\). Here we require \(X\) to be locally compact Hausdorff. Since \(K\) is compact, it may be covered by finitely many such \(J_x\), say by \(J_{x_1}, J_{x_2}, \ldots, J_{x_n}\). Consider the open sets \(S_i = S(J_{x_i}, M_{x_i}) \times S(J_{x_i}, N_{x_i})\) of the product topology on \(\mathcal{C}(X, Y) \times \mathcal{C}(X, Z)\). The element \((f, g)\) lies in \(S_i\) for all \(i\). Moreover, if \((f', g') \in \bigcap_{i=1}^n S_i\), then for any point \(x \in K\), the element \(x\) lies in \(J_{x_i}\) for some \(i\), then \((f(x_i), g(x_i)) \in M_{x_i} \times N_{x_i} \subseteq U\), so \((f', g') \in S\). Therefore we have found an open neighbourhood of \((f, g)\) contained in \(c^{-1}(S)\), and \(c\) is therefore continuous, as required.

(b) (This official solution contains more detail than was required on the homework). By the universal property of quotient spaces, and the presentation \(S^1 \approx I/0 \sim 1\), there is a natural bijection between functions \(\mathcal{C}_*(S^1, Y)\) and continuous functions \(\gamma : I \to Y\) such that \(\gamma(0) = \gamma(1) = y_0\), i.e., the based loops in \(Y\).

Consider two such based loops \(\gamma, \gamma'\), viewed as elements of \(\mathcal{C}_*(S^1, Y)\). Then there is a path from \(\gamma\) to \(\gamma'\) in this space if and only if there is a continuous function \(H : I \to \mathcal{C}_*(S^1, Y)\) such that \(H(0) = \gamma\) and \(H(1) = \gamma'\). But such a function induces a function we will call \(H' : I \to \mathcal{C}((S^1, Y)\), having the properties that \(H'(t)(s) = y_0\) for all \(t\) and that \(H'(0)(y) = \gamma(y)\) and \(H'(1)(y) = \gamma'(y)\) for all \(y \in I\). Since \(S^1\) is LCH (it is a closed subspace of \(\mathbb{R}^2\)), there is a bijection between the set of continuous functions \(I \to \mathcal{C}(S^1, Y)\) and the set of continuous functions \(\mathcal{C}(I \times S^1, Y)\). Therefore we have a continuous function \(H'' : I \times S^1 \to Y\), and this happens to have the additional properties that \(H''(t, s) = y_0\) for all values of \(t \in I\) and \(H''(0, y) = \gamma(y)\) and \(H''(1, y) = \gamma'(y)\).

We claim that \(I \times S^1\) is homeomorphic to the quotient of \(I \times I\) by the relation \((t, 0) \sim (t, 1)\) for all \(t \in I\).

It is easy to construct a continuous bijection \(\phi : I \times I \sim \to I \times S^1\), say by \(\phi(x, y) = (x, (\cos 2\pi y, \sin 2\pi y))\), and both source and target are compact Hausdorff, so this is a homeomorphism.
Therefore, by the universal property of quotients again, the function $H'' : I \times I \rightarrow Y$ is equivalent to a function $H' : I \times I \rightarrow Y$ with the following properties:

- $H(t, s) = y_0$ for all $t \in I$
- $H(0, y) = \gamma(y), H(1, y) = \gamma'(y)$ for all $y \in I$.

That is, $H$ is a homotopy, relative to $[0,1] \subset I$ from $\gamma$ to $\gamma'$.

In summary, $\mathcal{C}_* (S^1, Y)$ is canonically equivalent to the set of based loops in $(Y, y_0)$ and two loops $\gamma$ and $\gamma'$ are in the same path component of this space if and only if the corresponding loops are homotopic relative to endpoints. Therefore $\pi_0(\mathcal{C}_* (S^1, Y))$ is canonically in bijection with $\pi_1(Y, y_0)$.

(c)

**Lemma 0.1.** Let $Y$ and $Z$ be two topological spaces. The map $s : \pi_0(Y \times Z) \rightarrow \pi_0(Y) \times \pi_0(Z)$ induced by projections is a bijection.

**Proof.** Suppose $(y_0, z_0)$ and $(y_1, z_1)$ are two points in $Y \times Z$. They are in the same path component of $Y \times Z$ if and only if there is a continuous path $\mathcal{C}(I, Y \times Z)$ starting at one and ending at the other. The set of such continuous paths is in bijection (see the first part of this question) with pairs of continuous paths, one in $Y$ and one in $Z$ going from $y_0$ to $y_1$ and $z_0$ to $z_1$ respectively.

We show that $s$ is injective. Write $[x]$ for the path component of a point $x$ in the space containing it, which will be understood from the context. Suppose $s([y, z]) = s([y', z'])$, that is, $[y] = [y']$ and $[z] = [z']$. Then there is a path from $y$ to $y'$ in $Y$ and similarly for $z$ and $z'$. Therefore there is a path from $(y, z)$ to $(y', z')$, and so $[y, z] = [y', z']$.

We show $s$ is surjective: let $[y]$ be a path component of $Y$ and $[z]$ a path component of $Z$. Then $(y, z)$ is a point in $Y \times Z$ and $s([y, z]) = [y] \times [z]$.

Now we return to the question. Give $Y \times Z$ the evident basepoint $(y_0, z_0)$. Consider $\mathcal{C}_* (S^1, Y \times Z)$; this is a subset of $\mathcal{C}(S^1, Y \times Z)$, which is homeomorphic to $\mathcal{C}(S^1, Y) \times \mathcal{C}(S^1, Y)$ via a homeomorphism $s$ from the first part of this question. Under $s$, the subset $\mathcal{C}_* (S^1, Y \times Z)$ corresponds to those pairs of functions $\gamma : S^1 \rightarrow Y$ and $\delta : S^1 \rightarrow Z$ such that $\gamma(s) = y_0$ and $\delta(s) = z_0$, which is to say $s(\mathcal{C}_* (S^1, Y \times Z)) = \mathcal{C}_* (S^1, Y) \times \mathcal{C}_* (S^1, Z)$, and so $\pi_0(\mathcal{C}_* (S^1, Y \times Z))$ is in canonical bijection with $\pi_0(\mathcal{C}_* (S^1, Y)) \times \pi_0(\mathcal{C}_* (S^1, Z))$ by the lemma above. By the second part of this question, this is just the statement that $\pi_1(Y \times Z, (y_0, z_0))$ is in canonical bijection with $\pi_1(Y, y_0) \times \pi_1(Z, z_0)$.

□

4. Let $X$, $Y$ and $Z$ be topological spaces, and $f : X \rightarrow Y$, $f' : X \rightarrow Y$ and $h : Y \rightarrow Z$ be continuous functions. Suppose $f \simeq f'$.

(a) Prove that if $f$ is a homotopy equivalence, so too is $f'$;

(b) Prove that if any two of $f$, $h$ and $h \circ f$ are homotopy equivalences, then so is the third.
A quick way of doing this problem is as follows. There exists a category \( \text{Ho} \) having as objects, all topological spaces, and having as morphisms \( \text{Mor}(X, Y) \) the set of homotopy classes of continuous functions from \( X \) to \( Y \). We proved in Chapter 10 that composition of homotopy classes is well defined. It is immediate that composition with the homotopy classes of the identity functions is the identity on morphisms.

Now we know that \( f : X \rightarrow Y \) is a homotopy equivalence if and only if there exists \( g : Y \rightarrow X \) such that \( f \circ g \simeq \text{id}_Y \) and \( g \circ f \simeq \text{id}_X \). That is, writing \([f]\) for homotopy classes, if \([f \circ g] = [\text{id}_Y]\) and \([g \circ f] = [\text{id}_X]\). But \([f] \circ [g] = [f \circ g]\), and similarly in the other case, so we conclude that \( f \) is a homotopy equivalence if and only if \([f]\) is an isomorphism in \( \text{Ho} \).

(a) This is now immediate, since \([f] = [f']\), which is either an isomorphism or it isn’t.

(b) This is also easy, since given any two composable maps \( \phi, \psi \) in a category, if two out of three of \( \phi, \psi \) and \( \phi \circ \psi \) are isomorphisms, so is the third. In formulas

- \((\phi \circ \psi)^{-1} = \psi^{-1} \circ \phi^{-1}\)
- \(\phi^{-1} = \psi \circ (\phi \circ \psi)^{-1}\)
- \(\psi^{-1} = (\phi \circ \psi)^{-1} \circ \psi.\)

These all follow from the first formula, which is deduced by associativity of composition, so that \(\phi \circ \psi \circ \psi^{-1} \circ \phi^{-1}\) is an identity, as is \(\psi^{-1} \circ \phi^{-1} \circ \phi \circ \psi\). Then applying this observation with \(\phi = [f]\) and \(\psi = [h]\) gives the claim.

\[\Box\]

5. Let \( A_0, A_1 \) denote the two circles in \( \mathbb{R}^3 \) given by

\[A_0 = \{(x, y, 0) \mid x^2 + y^2 = 1\}, \quad A_1 = \{(x, y, 1) \mid x^2 + y^2 = 1\}.\]

Give \( \mathbb{R}^3 \) the basepoint \( x_0 = (0, 0, 0) \).

Consider \( X = \mathbb{R}^3 \setminus (A_0 \cup A_1) \). This is equipped with an inclusion map \( i : \mathbb{R}^3 \setminus (A_0 \cup A_1) \rightarrow \mathbb{R}^3 \setminus A_0 \).

(a) Using the van Kampen theorem, calculate \( \pi_1(\mathbb{R}^3 \setminus A_0, x_0) \).

(b) Using the van Kampen theorem, calculate \( \pi_1(X, x_0) \) and give a description of \( i_* \).

(c) Prove there is a loop in \( X \) that is not homotopic to a trivial loop, but becomes homotopic to a trivial loop both in \( \mathbb{R}^3 \setminus A_0 \) and in \( \mathbb{R}^3 \setminus A_1 \).

(a) Write \( Y = \mathbb{R}^3 \setminus A_0 \)

There are several ways of doing this. The answer is that \( \pi_1(Y, x_0) \cong \mathbb{Z} \), with a generator being given by a loop around the missing \( A_0 \) as illustrated.

Two methods are given. In each case, begin by observing that there is a deformation retract of \( Y \) onto the complement of \( A_0 \) in a compact convex \( K \) set containing \( A_0 \), as illustrated.
Split $K \setminus A_0$ in two open sets, $U$ and $V$ as illustrated. Each deformation retracts onto a slightly smaller closed subset of $K$, say $U' = U \setminus V$ and $V' = V \setminus U$. The sets $U'$, $V'$ are homeomorphic to a closed ball in $\mathbb{R}^3$ missing a diameter.

But this deformation retracts onto a closed disk in $\mathbb{R}^2$ less a point, which deformation retracts onto $S^1$. So $\pi_1(U, x_0) = \pi_1(V, x_0) = Z \times Z$. In each case a generator is given by a loop going around $A_0$ once.

Then the intersection $U \cap V$ has $S^1 \vee S^1$ as a deformation retract, as illustrated in Figure 4. We have $\pi_1(U \cap V, x_0) = F_2 = \langle \xi, \eta \rangle$ and the map $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ sends both $\xi$ and $\eta$ to a generator, and similarly for $V$. So the amalgamated product is the quotient of the free product $\ast$ by the relation $u = v$. That is, $\pi_1(Y, x_0) = Z$, generated by the class of a loop around $A_0$.

The space $K \setminus A_0$ deformation retracts (by fattening $A_0$ until it consists of the interior of a torus, then pushing $K$ down until it is reduced to just a 2 dimensional surface, rather than a solid), onto a space $M$ which is the union of a torus $T$ and a disk $D$ used to fill in the doughnut hole of the torus.

Enlarge $D$ to $D'$ by finding an open neighbourhood around it in $T \cup D$, and similarly for $T'$, making $T'$. Choose a path from $x_0$ (which lies in the interior of $D$) to a point in $T' \cap D'$. These spaces are illustrated in Figures 5–8.

The space $T' \cap D'$ deformation retracts onto $S^1$. We can now apply the van Kampen theorem with $T'$ and $D'$ giving $\pi_1(X, x_0) \cong \pi_1(T', x_0) \ast_{\pi_1(S^1, x_0)} \pi_1(D', x_0)$ which is $Z^2 \ast Z$. If we name two generators for $\pi_1(T', x_0)$ as say $\xi$ (going around the tube of the torus) and $\eta$ (going around the doughnut hole of the torus), then the effect of gluing on $D'$ is to annihilate $\eta$. Again we
deduce that \( \pi_1(Y, x_0) \cong \mathbb{Z} \), generated by the class of a loop around \( A_0 \).

(b) It is helpful to nudge \( A_0 \) down (or \( x_0 \) up), then divide \( \mathbb{R}^3 \) into two open half-spaces, each containing exactly one of \( A_0 \) or \( A_1 \), and intersecting in a convex set. Then the van Kampen theorem applies to say \( \pi_1(\mathbb{R}^2 \setminus (A_0 \cup A_1), x_0) = \mathbb{Z} \xi * \mathbb{Z} \eta \), since the intersection is simply connected. That is, the fundamental group is free on two generators, each representing a loop around one of the two missing circles, as illustrated in Figure 9. The map \( i_* \) here amounts to filling in the circle \( A_1 \), so is the map \( \mathbb{Z} \xi * \mathbb{Z} \eta \to \mathbb{Z} \xi \) given by evaluating \( \eta \) at the trivial element, since \( \eta \) becomes contractible when \( A_1 \) is not removed.

(c) The homotopy class of any loop in \( X \) is a product of powers of noncommuting group elements \( \xi \) and \( \eta \). The effect of filling in either \( A_0 \) or \( A_1 \) is to evaluate at \( \xi = e \) or \( \eta = e \) respectively. Therefore it suffices to find a nontrivial word in \( \xi \) and \( \eta \) that becomes trivial after either evaluation, for instance \( \xi \eta \xi^{-1} \eta^{-1} \). Any loop in this homotopy class will suffice. One example is given in Figure 10. There are many other homotopy classes that also work, for instance \( \xi^{-1} \eta^{-1} \xi^2 \eta^2 \xi^{-1} \eta^{-1} \).

\[ \square \]
Figure 3: $U \setminus V$ is homeomorphic to a ball less a diameter

Figure 4: The space $U \cap V$, and $S^1 \vee S^1$ as a deformation retract.

Figure 5: The space $M$. 
Figure 6: The space $T'$.  

Figure 7: The space $D'$.  

Figure 8: The space $T' \cap D'$.  

Figure 9: Generators for $\pi_1(X, x_0)$
Figure 10: A loop in the class of $\xi \eta^{-1} \eta^{-1}$