1. Let $X$ be a topological space. A $G_δ$ set in $X$ is a set that can be expressed as a countable intersection of open subsets in $X$. Let $f : X → M$ be a function (not necessarily continuous) where $M$ is a metric space. For any $n \in \mathbb{N}$ and any point $x \in X$, say that $x \in C(f, n)$ if there exists some open set $U_n \ni x$ such that $d(f(x_1), f(x_2)) < 1/n$ for all $x_1, x_2 \in U_n$. Observe that if $x \in C(f, n)$, then there is an open neighbourhood $x \in U_n$ such that $U_n \subseteq C(f, n)$. Therefore, $C(f, n) \subseteq X$ is open

(a) Consider $Y = \bigcap_n C(f, n)$. This is a $G_δ$ subset of $X$. Prove that $f|_Y : Y → M$ is continuous.

(b) Suppose $X$ is also metric (so we can define continuity “at a point of $x$”), and that $x \in X \setminus Y$. Prove that $f$ is not continuous at $x$. Hint: there is some $n$ such that $x \notin C(f, n)$. Then try to find a $δ$ corresponding to $ε = 1/(2n)$ in an $ε$-$δ$ characterization of continuity at $x$.

(c) Prove that $\mathbb{Q}$ is not a $G_δ$ subset of $\mathbb{R}$ with the usual topology (use the Baire category theorem).

2. This question requires some knowledge of $\mathbb{C}$, the field of complex numbers. The field $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ as an $\mathbb{R}$ vector space, and we give it the usual $d_j$ metric topology. You may assume that the field operations of addition, multiplication and division are continuous in $\mathbb{C}$.

Fix an integer $n ≥ 1$. Define a space, the complex projective space of dimension $n$, $\mathbb{C}P^n$ as the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ by the relation

$$(z_0, z_1, \ldots, z_n) \sim (λz_0, λz_1, \ldots, λz_n),$$

where $λ \in \mathbb{C} \setminus \{0\}$. That is, there is a surjective continuous function

$$q : \mathbb{C}^{n+1} \setminus \{0\} → \mathbb{C}P^n.$$  

Write the class of $(z_0, z_1, \ldots, z_n)$ in $\mathbb{C}P^n$ as $[z_0; z_1; \ldots; z_n] = q(z_0, \ldots, z_n)$.

(a) For $j \in \{0, \ldots, n+1\}$, write $V_j$ for the subspace of $\mathbb{C}^{n+1}$ given by the condition $z_j = 0$. Write $U_j$ for the image of $q(\mathbb{C}^{n+1} \setminus V_j) ∈ \mathbb{C}P^n$. Prove $U_j$ is open in $\mathbb{C}P^n$.

(b) Consider the function $f : U_0 → \mathbb{C}P^n$ given by $f([z_0; z_1; \ldots; z_n]) = (z_1/z_0, z_2/z_0, \ldots, z_n/z_0)$. Prove that $f$ is a homeomorphism—i.e., show it is continuous and has a continuous inverse.

Let $A ∈ \text{GL}_{n+1}(\mathbb{C})$ be an invertible $(n+1) × (n+1)$ matrix. The transformation $T_A(\mathbf{v}) ↔ (A\mathbf{v})^T$ is a continuous map $\mathbb{C}^{n+1} \setminus \{0\} → \mathbb{C}^{n+1} \setminus \{0\}$. Since the transformation $\mathbf{v} ↔ T_A\mathbf{v}$ satisfies $T_A(λ\mathbf{v}) = λT_A\mathbf{v}$, it follows from the universal property of the quotient topology that it descends to a continuous function $T_A : \mathbb{C}P^n → \mathbb{C}P^n$. You may assume that $T_A \circ T_B = T_{AB}$, and that $T_{[I_{n+1}]} = \text{id}_{\mathbb{C}P^n}$, so that the $T_A$ are homeomorphisms $T_A : \mathbb{C}P^n → \mathbb{C}P^n$.

(c) Prove that $\mathbb{C}P^n$ is Hausdorff.
(d) By considering the set of complex numbers $S_{n+1} \subset \mathbb{C}^{n+1}$ given by $|z_0|^2 + \cdots + |z_n|^2 = 1$, prove that $\mathbb{CP}^n$ is compact.

(e) Prove that $\mathbb{C}^n \approx U_0 \subset \mathbb{CP}^n$ is a compactification of $\mathbb{C}^n$.

3. Given a space $X$, let $\pi_0(X)$ denote the set of path components of $X$. Given a map $f : X \to Y$, define $f_* : \pi_0(X) \to \pi_0(Y)$ by sending the path component of $x \in X$ to the path component of $f(x) \in Y$. You may assume that $\pi_0$ is a functor from $\textbf{Top}$ to $\textbf{Set}$, the category of sets and functions.

(a) Suppose $X$ is Hausdorff and $A$ is a retract of $X$. Show $A$ is closed in $X$.

(b) Show that if $A$ is a retract of $X$, then the canonical map $\pi_0(A) \to \pi_0(X)$ is an injection.

(c) Embed $\mathbb{R} \setminus \{0\}$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ as the $x$-axis (minus the origin). Show that $\mathbb{R} \setminus \{0\}$ is not a retract of $\mathbb{R}^2 \setminus \{(0,0)\}$.

4. Let $S^n$ denote the subset of $\mathbb{R}^{n+1}$ consisting of those points $x = (x_1, \ldots, x_{n+1})$ such that $\sum_{i=1}^{n+1} x_i^2 = 1$.

(a) Prove that the inclusions $i : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ are homotopy equivalences for $n = 0, 1, 2$.

(b) Prove that no two of the following three spaces are homeomorphic: $S^0$, $S^1$, $S^2$.

(c) Prove that $S^2$ is not homeomorphic to the closed unit disk: $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

5. Suppose $X$ is a topological space. Let us say $x \sim y$ if, for any continuous function $f : X \to \{0,1\}$ with $f(x) = 0$, we also have $f(y) = 0$. The relation $\sim$ is easily seen to be an equivalence relation. The equivalence classes of points under $\sim$ are called the \textit{quasicomponents} of $X$.

(a) Prove the quasicomponents of $X$ are closed sets, and each connected component is contained in a quasicomponent.

(b) Prove that if $X$ is locally connected, then the quasicomponents of $X$ agree with the connected components.

(c) Let $X$ be the following subspace of $\mathbb{R}^2$:

\[ \left\{ \left( \frac{1}{n}, y \right) \in \mathbb{R}^2 : n \in \mathbb{N}, y \in [-1,1] \right\} \cup \{(0,0), (0,1)\}. \]

Describe the quasicomponents.

(d) Let $Y$ denote the union of the space $X$ in the previous part with the coordinate axes. Prove that $Y$ is path connected but not locally connected.