1. Let \((X, d)\) be a metric space. Give \(X \times X\) the product topology. Prove that the function \(d : X \times X \to \mathbb{R}\) is continuous.

2. If \(R\) is a commutative ring, we define \(\text{Spec } R\) to be the set of prime ideals of \(R\). For \(f \in R\), we define \(U_f = \{p \in \text{Spec } R \mid f \not\in p\}\), and let \(\mathcal{Z} = \{U_f \mid f \in R\}\). The topology generated by \(\mathcal{Z}\) is the Zariski topology.

   (a) Prove that \(\mathcal{Z}\) is closed under finite intersections—by an easy induction argument it is sufficient to prove it is closed under intersections \(U_f \cap U_g\). The set \(\mathcal{Z}\) therefore is a basis for the Zariski topology (not just a subbasis).

   (b) Let \(R = \mathbb{Z}\). Describe \(\text{Spec } \mathbb{Z}\) and \(U_7\).

   (c) Describe, with proof, the closed sets of the Zariski topology on \(\text{Spec } \mathbb{Z}\).

   (d) Show that the Zariski topology on \(\text{Spec } \mathbb{Z}\) is \(T_0\) but not \(T_1\).

3. Define a topology on \(\mathbb{R}\) by specifying that the open intervals \((a, b)\) are open, and that also the set \(U = \mathbb{R} - \{\frac{1}{a} \mid a \in \mathbb{Z} - \{0\}\}\) is open—i.e. the topology generated by the open intervals and \(U\). Prove that this topology is Hausdorff but not regular.

4. This problem proves some important facts about the product topology that were left unproved in the class notes. Let \(\{X_i\}_{i \in I}\) be a family of topological spaces, let \(\prod_{i \in I} X_i\) denote the product space and let \(\pi_i : \prod_{i \in I} X_i \to X\) denote the projection maps. If \(\{A_i \subseteq X_i\}_{i \in I}\) are subsets, one in each \(X_i\), then \(\prod_{i \in I} A_i\) is used to denote \(\bigcap_{i \in I} \pi_i^{-1}(A_i)\).

   Recall that the product topology has a basis consisting of sets \(\prod_{i \in I} U_i\) where each \(U_i \subset X_i\) is open, and all but finitely many of the \(U_i\) are actually equal to \(X_i\)—equivalently, a basis consisting of sets \(\bigcap_{\text{finite}} \pi_i^{-1}(U_i)\).

   (a) Prove that the projection maps \(\pi_i\) are open maps.

   (b) Suppose \(\{A_i \subseteq X_i\}\) are closed sets. Prove that \(\prod_{i \in I} A_i\) is a closed subset of \(\prod_{i \in I} X_i\). Hint: suppose \(x \not\in \prod_{i \in I} A_i\), then there is at least one \(i\) such that \(\pi_i(x) \not\in A_i\).

   (c) Prove that for any family of subsets \(\{A_i \subseteq X_i\}\),

   \[
   \prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}
   \]

   (d) Give an example to show that the analogous statement is false when the closure is replaced by the interior.

   (e) Suppose each \(X_i\) is nonempty. For any given \(i\), construct, with proof, a continuous function \(s_i : X_i \to \prod_{i \in I} X_i\) such that \(\pi_i \circ s_i = \text{id}_{X_i}\)—the identity map on the topological space \(X_i\).

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Footnote:

1A prime ideal \(p \subset R\) is an ideal of \(R\) such that \(fg \in p\) implies either \(f \in p\) or \(g \in p\).
5. Read section 4 of the notes on $p$-norms. Let $\mathbb{R}^\mathbb{N}$ denote the space $\prod_{i=1}^{\infty} \mathbb{R}$ with the product topology, and let $b$ denote this space with the box topology. Let $\ell^\infty$ denote the space of bounded sequences, with the topology induced by the $\|\cdot\|_\infty$ norm, so that $d_\infty((x_n), (y_n)) = \sup_{n \in \mathbb{N}} |x_n - y_n|$. The metric topology for $\|\cdot\|_\infty$ norm is called the uniform topology.

(a) Observe that $\ell^\infty \subset \prod_{i=1}^{\infty} \mathbb{R}$ as sets. Define elements of $X_m \in \ell^\infty$ as follows:

$$X_m = (1,1,\ldots,1,0,0,0\ldots) .$$

Prove that $(X_m)$ converges in the product topology but not in the box topology or in the uniform topology—be careful, this is a sequence of sequences, so

$$X_1 = (1,0,0,\ldots)$$
$$X_2 = (1,1,0,\ldots)$$
$$\vdots$$
$$X_m = (1,1,\ldots,1,0,\ldots)$$
$$\vdots$$

(b) Define elements $Y_m \in \ell^\infty$ by

$$Y_m = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \ldots \right).$$

Prove the sequence $(Y_m)$ converges in the product topology and in the uniform topology, but not in the box topology.