1. Read the supplementary notes on $p$-norms.

(a) Show that for a given vector $x \in \mathbb{R}^n$, the function $p \mapsto \|x\|_p$ is (weakly) decreasing on $p \in [1, \infty]$.

(b) Show that $\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$.

(a) The quantity $\|x\|_p$ does not change if we replace $x$ by $(|x_1|, \ldots, |x_n|)$, so we may assume each coordinate of $x$ is nonnegative. If $x = 0$, the result is trivial, so we assume $x \neq 0$. Note that $\|Ax\|_p = \|A\| \|x\|_p$, so we may scale $x$ so that $\|x\|_p = 1$. Then

$$1 = \|x\|_p^p = \sum_{i=1}^{n} x_i^p$$

where each $x_i$ is a real number satisfying $0 \leq x_i \leq 1$. Since $f(q) = x^q$ is decreasing for such $x$, it follows that

$$1 = \|x\|_p^p \geq \|x\|_q^q$$

whenever $q < p$. It follows that

$$1 = \|x\|_p \geq \|x\|_q.$$

Note that if $\|x\|_p = 1$ then the maximum value a coordinate of $x$ may assume is 1, so it follows that $\|x\|_p \geq \|x\|_\infty$ as well.

(b) As above, without loss of generality we may assume that in $x = (x_1, \ldots, x_n)$, each of the $x_i$ is nonnegative and at least one is positive. Rescale the vector so that $\|x\|_\infty = 1$, so at least one coordinate takes the value 1. But for any $p$

$$1 \leq \|x\|_p^p = \sum_{i=1}^{n} x_i^p \leq n$$

So that

$$1 \leq \|x\|_p \leq n^{1/p}$$

where $n$ is a positive integer. Taking $\lim_{p \to \infty}$ and applying the squeeze principle gives

$$\lim_{p \to \infty} \|x\|_p = 1 = \|x\|_\infty.$$
2. Again, refer to the supplementary notes on $p$-norms.

(a) Suppose $((X_1, d_1), \ldots, (X_n, d_n))$ is a finite set of metric spaces and write $X = \prod_{i=1}^n X_i$. Let $p \in [0, \infty]$. Define a product $p$-metric on $X$ as follows. If $x = (x_1, \ldots, x_n)$ and $y \in (y_1, \ldots, y_n)$ are elements of $X$, then let

$$d(x, y) = \|(d_1(x_1, y_1), d_2(x_2, y_2), \ldots, d_n(x_n, y_n))\|_p.$$ 

Prove that $d$ is a metric on $X$.

(b) Prove that $d_p(x, y) = \|x - y\|_p$ defines a metric on $\mathbb{R}^n$ for any $p \in [1, \infty]$. You can do this directly, or apply the previous part to an $n$-fold product of $(\mathbb{R}, |\cdot|)$.

(c) Let $p, q \in [1, \infty]$. Show that the metric topology associated to $d_p$ and to $d_q$ on $\mathbb{R}^n$ agree. It may be helpful to refer to Lemma 20.2 of Munkres’ Topology.

(a) Symmetry and the property that $d(x, y) = 0$ if and only if $x = y$ are easily proved. What remains is the triangle inequality. Let $x, y$ and $z$ be three vectors.

$$d(x, z) + d(x, z) = \|(d_1(x_1, z_1), d_2(x_2, z_2), \ldots, d_n(x_n, z_n))\|_p + \|(d_1(y_1, z_1), d_2(y_2, z_2), \ldots, d_n(y_n, z_n))\|_p$$

$$\geq \|(d_1(x_1, z_1) + d_1(y_1, z_1), d_2(x_2, z_2) + d_2(y_2, z_2), \ldots, d_n(x_n, z_n) + d_n(y_n, z_n))\|_p$$

by the Minkowski inequality from the supplementary notes. Now, note that $\|x_1, \ldots, x_n\|_p$ is an increasing function of each $x_i$ individually, so by the triangle inequality for each $d_i$, we see that

$$\|(d_1(x_1, z_1) + d_1(y_1, z_1), d_2(x_2, z_2) + d_2(y_2, z_2), \ldots, d_n(x_n, z_n) + d_n(y_n, z_n))\|_p$$

$$\geq \|(d_1(x_1, y_1), d_2(x_2, y_2), \ldots, d_n(x_n, y_n))\|_p = d(x, y).$$

This proves the triangle inequality.

(b) We observe that applying the previous part to $n$ copies of $(\mathbb{R}, |\cdot|)$ endows the space $\mathbb{R}^n$ with the metric

$$d(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p} = \|x - y\|_p$$

which is the required metric.

(c) It’s sufficient to show this when $q = 1$ and $p$ is arbitrary. By the supplementary notes on $p$-norms, we know that for any $x$, there is an inequality

$$\|x\|_1 \geq \|x\|_p \geq \frac{1}{n} \|x\|_1.$$ 

In particular, for any $x \in \mathbb{R}^n$ and any $\epsilon > 0$, we have $B_p(x, \epsilon) \subset B_1(x, \epsilon)$, so that the topology associated to the $p$-norm is finer than that associated to the 1-norm. But we also have $B_1(x, \epsilon) \subset B_p(x, n\epsilon)$, showing the converse. Therefore the two topologies agree.

\[\square\]
3. Let $C$ be a category and let $f : A \to B$ be a morphism in $C$. We say $f$ is an epimorphism if it has the property that whenever $g_1 : B \to X$ and $g_2 : B \to X$ are morphisms such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$. Describe, with proof, the epimorphisms in the category of sets.

(a) We claim the epimorphisms are exactly the surjective maps.

First we show a surjective map $f : A \to B$ is an epimorphism. Suppose given a surjective map $f : A \to B$ and two maps $g_1 : B \to X$ as in the question. Then for any $b \in B$, there exists $a \in A$ such that $f(a) = b$, and then $g_1(b) = (g_1 \circ f)(a) = (g_2 \circ f)(a) = g_2(b)$. Since $b$ was arbitrarily chosen, it follows that $g_1 = g_2$.

Now suppose given an epimorphism $f : A \to B$ in the category of sets. Define $g_1 : B \to \{0,1\}$ as $g_1(b) = 0$ if $b \in \text{im} f$ and $g_1(b) = 1$ otherwise. Define $g_2 : B \to \{0,1\}$ as the constant function 0. Note that for all $a \in A$, we have $g_i(f(a)) = 0$ for both $i = 1$ and $i = 2$. Therefore it must be the case that $g_1 = g_2$, so that every element of $B$ must lie in the image of $f$.

4. Give $\mathbb{N}$ the cofinite topology, where the nonempty open sets are those sets containing all but finitely many elements. Give $\mathbb{R}$ the usual topology. Describe, with proof, the continuous functions $f : \mathbb{N} \to \mathbb{R}$.

Suppose $f$ is a constant function with value $r$. Then $f$ is continuous since $f^{-1}(U)$ for any open set $U$ is either $\mathbb{N}$ or $\emptyset$, depending on whether $r \in U$.

Suppose $f$ continuous but is not constant. There are integers $m$ and $n$ satisfying $f(m) < f(n)$. Let $\epsilon = (f(n) - f(m))/2$. Then $f^{-1}((n - \epsilon, n + \epsilon))$ and $f^{-1}((m - \epsilon, m + \epsilon))$ are disjoint nonempty open subsets of $\mathbb{R}$, a contradiction.

5. Let $\{(A_i, \tau_i)\}_{i \in I}$ be a family of topological spaces. Let $A = \prod_{i \in I} A_i$ and let $\tau$ denote the product topology. Let $U$ be a nonempty open set and denote $\text{im}(\pi_i|_U)$ by $\pi_i(U)$. Note that

$$\pi_i(U) = A_i$$

for all but finitely many values of $i \in I$ (we say for almost all values of $i$). This can be proved by proving it for the elements of the basis consisting of finite intersections of sets of the form $\pi_i^{-1}(U)$.
(a) Define a topology, $\beta$, on $A = \prod_{i \in I} A_i$ as the topology generated by sets of the form

$$\bigcap_{i \in I} \pi_i^{-1}(V_i)$$

where $V_i$ is open in $A_i$ for all values of $i \in I$. This is the box topology$^1$. Prove that the box topology is finer than the product topology.

(b) Prove that if $I$ is finite, the box topology and product topology are the same.

(c) Prove that if $I = \mathbb{N}$ and $A_i = \mathbb{R}$ with the usual topology that the box and product topologies are not the same. It may be helpful here, and subsequently, to understand this product $\prod_{i=1}^\infty \mathbb{R}$ as the set of all infinite sequences of real numbers.

(a) It is sufficient to show that the sets making up a subbasis for the product topology are open in the box topology. One such subbasis consists of sets of the form $\pi_i^{-1}(U)$ where $U$ is open in $A_i$, and these sets satisfy

$$\pi_j \pi_i^{-1}(U) = \begin{cases} U, & \text{if } i = j; \\ A_j, & \text{otherwise.} \end{cases}$$

and they are therefore open for the box topology, as required.

(b) It suffices to show that a subbasis for the box topology consists of open sets for the product topology. The obvious choice consists of sets of the form $U_1 \times \ldots \times U_n$ where each $U_i$ is open in $A_i$. But this is the set

$$\bigcap_{i=1}^n \pi_i^{-1}(U_i)$$

which is open in the product topology.

(c) As remarked in the question, if $U$ is open in the product topology then $\pi_i(U) = A_i$ for almost all $i$. But the set $I = \prod_{i=1}^\infty (0, 1) \subset \prod_{i=1}^\infty \mathbb{R}$, that is, the subset consisting of sequences $(r_1, r_2, \ldots)$ such that $0 < r_i < 1$ for all $i$, does not satisfy $\pi_i(I) = \mathbb{R}$ for any $i$ (let alone almost all), and it is open in the box topology.

6. Let $\mathbb{R}^\mathbb{N}$ denote the set $\prod_{i=1}^\infty \mathbb{R}$ (i.e. the set of all sequences $(r_1, r_2, \ldots)$ of real numbers), and let $c_{00}^2$ denote the subset consisting of those sequences $(r_1, r_2, \ldots)$ that are eventually 0, that is, for which there exists $N \in \mathbb{N}$ such that $r_i = 0$ for $i > N$.

(a) Read Theorem 19.1 of Munkres' topology.

$^1$In a previous version of this homework, there was a mistake in the definition of the box topology. This has been fixed.
(b) Give $\mathbb{R}^\mathbb{N}$ the product topology. Show that $c_{00} = \mathbb{R}^\mathbb{N}$.

(c) Give $\mathbb{R}^\mathbb{N}$ the box topology. What is $c_{00}$?

(a) Do this.

(b) Let $x = (x_1, x_2, \ldots) \in \mathbb{R}^\mathbb{N}$ be a sequence, and let $V \ni x$ be a neighbourhood of $x$. Then $V$ contains a basis element containing $x$, i.e. a set of the form $U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R} \times \mathbb{R} \times \cdots$ where each $U_i$ is open in $\mathbb{R}$. Since $x_i \in U_i$, no $U_i$ can be empty. We may choose a sequence $r = (r_1, r_2, \ldots, r_n, 0, 0, 0, \ldots)$ where $r_i \in U_i$. Then $r \in V \cap c_{00}$, and so every open neighbourhood of $x$ has nonempty intersection with $c_{00}$, therefore $x \in c_{00}$.

(c) Suppose $x \notin c_{00}$. Construct an open neighbourhood as follows: if $x_i = 0$, then let $U_i = \mathbb{R}$. If $x_i \neq 0$, then let $U_i \subset \mathbb{R}$ be an open neighbourhood of $x_i$ that does not contain 0. Since $x \notin c_{00}$, we must have $0 \notin U_i$ for infinitely many values of $i$. Then let $V = U_1 \times U_2 \times \cdots$, which is an open neighbourhood of $x$ in the box topology. This neighbourhood does not contain any sequence that is eventually 0 since there are infinitely many $U_j \neq 0$, so $x \notin c_{00}$. It follows that $c_{00} = c_{00}$.

\[\square\]

\[^{2}\text{After having written this homework, I realized I would want the notation } c_0 \text{ for a different set, the set of sequences converging to 0. So the notation here has been changed to } c_{00}.\]