1. Some miscellaneous questions about metric spaces. Let \((X,d)\) be a metric space.

(a) Let \(A \subseteq X\) be a nonempty subset. Define \(\text{dist}_A(x) : X \to [0,\infty)\) by \(\text{dist}_A(x) = \inf_{a \in A} d(x, a)\). Prove that \(\text{dist}_A(x)\) is continuous—an \(\epsilon - \delta\) argument may be easier here.

(b) Let \(A \subseteq X\) be a subset. Prove that the following are equal: \(\tilde{A}\) and \(\text{dist}^{-1}_A(0)\).

(c) Let \(C_1\) and \(C_2\) be disjoint closed subsets. Prove there exist disjoint open sets \(U_1 \supseteq C_1\) and \(U_2 \supseteq C_2\).

(d) Suppose \(W\) is a dense subset of \(X\). Prove that for every open \(U\) and every \(x \in U\), there is some ball \(B(w, 1/n)\) where \(n\) is a natural number such that \(x \in B(w, 1/n) \subseteq U\).

(a) First observe that since we have defined \(\text{dist}_A(x)\) as the infimum of a nonempty subset of \(\mathbb{R}\), there exists a sequence of entries \(a_n \in A\) such that \(\text{dist}_A(x) = \lim_{n \to \infty} d(x, a_n)\). For any two points, \(x, y \in X\), we have

\[
\text{dist}_A(y) = \inf_{a \in A} d(y, a) \leq \inf_{a_n} d(y, a_n) \leq \inf_{a_n} d(y, a_n) + d(x, a_n) = d(y, x) + \text{dist}_A(x)
\]

so that

\[
\text{dist}_A(y) - \text{dist}_A(x) \leq d(x, y)
\]

and by symmetry

\[
|\text{dist}_A(x) - \text{dist}_A(y)| \leq d(x, y).
\]

Therefore, if \(\epsilon > 0\), we can take \(\delta = \epsilon\) and deduce that if \(d(x, y) < \delta\), then \(|\text{dist}_A(x) - \text{dist}_A(y)| < \epsilon\). This proves that \(\text{dist}_A\) is a continuous function.

(b) Clearly, \(\text{dist}^{-1}_A(0)\) is a closed subset of \(X\) containing \(A\). Therefore it contains \(\tilde{A}\). Conversely, suppose \(x \in \tilde{A}\), then there exists a sequence \(a_n \in A\) such that \(a_n \to x\). But then \(\text{dist}_A(x) = \lim_{n \to \infty} d(a_n, x)\) for all \(n\), and since the right hand side tends to 0, it follows that \(\text{dist}_A(x) = 0\).

(c) Consider the function \(f(x) = \text{dist}_{C_1}(x) - \text{dist}_{C_2}(x)\). This is continuous, being the difference of two continuous functions. Let \(U_1 = f^{-1}((-\infty, 0))\) and \(U_2 = f^{-1}((0, \infty))\). These are two disjoint open subsets of \(X\). We claim that \(C_1 \subseteq U_1\). To see this, observe that, since \(C_2 \cap C_1 = \emptyset\) by hypothesis, no point in \(C_1\) satisfies \(\text{dist}_{C_2}(x) \leq 0\). On the other hand, all points \(x \in C_1\) satisfy \(\text{dist}_{C_2}(x) = 0\). Therefore, for points \(x \in C_1\), we see that \(f(x) < 0\), so \(x \in U_1\). The argument for \(U_2\) is symmetric.

(d) Let \(x\) be a point and \(U \ni x\) an open neighbourhood. Let \(B(x, 1/m)\) be an open ball around \(x\) contained in \(U\). Then \(B(x, 1/(2m))\) is an open ball as well. Any open subset of \(X\) contains a
point \( w \in W \). Let \( w \in B(x, 1/(2m)) \) and consider \( B(w, 1/(2m)) \). We know that \( d(w, x) < 1/(2m) \) so this ball contains \( x \). Also, if \( y \in B(w, 1/(2m)) \) then the triangle inequality tells us that \( d(x, y) \leq 1/(2m) + 1/(2m) = 1/m \), so that \( y \in B(x, 1/m) \subseteq U \). Therefore \( B(w, 1/(2m)) \subseteq U \) as required.

\[ \square \]

2. Give \( \mathbb{R} \) the \textit{counicalable} topology \( \tau \). The closed sets are \( \mathbb{R} \) itself and the countable subsets, i.e., all images of functions \( f : \mathbb{N} \rightarrow \mathbb{R} \). You may assume without proof that the complement of any countable subset of \( \mathbb{R} \) is infinite (in fact, is not a countable set).

(a) Describe all convergent sequences in \( (\mathbb{R}, \tau) \).

(b) Describe all sequentially closed subsets of \( (\mathbb{R}, \tau) \).

(c) Prove \( (\mathbb{R}, \tau) \) is not first countable.

(a) Let \( x_n \rightarrow x \) be a convergent sequence. We claim that there exists some \( N \in \mathbb{N} \) such that \( x_n = x \) for all \( n \geq N \). If not, then we can find some subsequence \( (x_{n_i}) \) of \( (x_n) \) such that \( x_{n_i} \neq x \) for all \( i \) and \( x_{n_i} \rightarrow x \) since subsequences of convergent sequences converge. But now consider \( \mathbb{R} \setminus \{x_{n_i}\}_{i=1}^{\infty} \). This is an open neighbourhood of \( x \) in the cocountable topology that contains no term of the sequence \( x_{n_i} \), which allegedly converges to \( x \). This is a contradiction.

In the other direction, any eventually-constant sequence converges in all topologies for trivial reasons.

(b) If a sequence \( x_n \) in \( A \) converges to some \( x \in \mathbb{R} \), then \( x_n \) is eventually constant, so that \( x \in A \). Therefore, all subsets of \( \mathbb{R} \) are sequentially closed.

(c) Let \( A \) be an uncountable proper subset of \( \mathbb{R} \), say \( A = [0, 1] \). Then \( A \) is sequentially closed in the cocountable topology, but it is not closed. Since sequentially closed sets in first-countable topologies are closed, it follows that the cocountable topology is not first countable.

\[ \square \]

3. Let \( \mathbb{R}^n \) be given the usual (metric) topology, and let \( S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \) be given the subspace topology as a subset of \( \mathbb{R}^2 \). Consider the function \( f : \mathbb{R} \rightarrow S^1 \) given by \( f(\theta) = (\cos 2\pi \theta, \sin 2\pi \theta) \). You may assume \( f \) is continuous. Let \( H = H_{x>0} \) denote the subset of \( S^1 \) consisting of points where \( x > 0 \). Define \( H_{x<0}, H_{y>0} \) and \( H_{y<0} \) similarly.

(a) Give an open interval \( I \subset \mathbb{R} \) and a continuous function \( g : \{(x, y) \in \mathbb{R}^2 \mid -1 < y < 1\} \rightarrow \mathbb{R} \) such that \( f(I) = H \) and \( g(H) = I \), and that the restricted map \( f|_I : I \rightarrow H \) is a homeomorphism with inverse \( g|_H : H \rightarrow I \). You may assume any differentiable function \( h : U \rightarrow \mathbb{R} \), where \( U \) is an open interval, is continuous.
(b) Give, without proof, similar intervals and functions $g$ for $H_{x<0}$, $H_{y>0}$ and $H_{y<0}$.

(c) Show that $f$ is an open function.

(d) Deduce that the map $h : [0,1]/\{0,1\} \to S^1$ given by $h(\theta) = (\cos 2\pi \theta, \sin 2\pi \theta)$ is a homeomorphism.

(a) Consider the function $g : ((x, y) \in \mathbb{R}^2 \mid -1 < y < 1) \to \mathbb{R}$ given by $g((x, y)) = \arcsin y$. This is the composite of the two continuous functions $\pi_2 : ((x, y) \in \mathbb{R}^2 \mid -1 < y < 1) \to (-\pi/2, \pi/2)$ given by projection on the second variable and the differentiable function $\arcsin : (-1, 1) \to \mathbb{R}$.

Let $I = (-\pi/2, \pi/2)$. For $\theta \in I$, we have $\cos \theta > 0$, so $f(I) \subset H$. Moreover, $g(H) = \arcsin y \in I$, using the usual definition of $\arcsin$. Both $f$, $g$ are continuous, $g \circ f = \arcsin \sin$, which is the identity on $I$, and $f \circ g(x, y) = (\sqrt{1 - y^2}, y) = (x, y)$ if $(x, y) \in H_{x>0}$. It follows that $f|_I : I \to H$ is a homeomorphism.

(b) There are many possible choices. The following represent a particular solution. For $H_{x<0}$, choose $g((x, y)) = \pi - \arcsin y$, with $I = (\pi/2, 3\pi/2)$. For $H_{y>0}$ choose $g((x, y)) = \arccos x$ and $I = (0, \pi)$. For $H_{y<0}$ choose $g((x, y)) = -\arccos x$ and $I = (-\pi, 0)$.

(c) It suffices to show that there is a basis of the topology on $\mathbb{R}$ such that $f(U)$ is open for all basis elements $U$. One basis for $\mathbb{R}$ is given by the set of open intervals of length $\ell < \pi/2$. Any such open interval $(a, b)$ lies entirely within an open interval of one of the following four forms:

i. $I = ((2n - 1/2)\pi, (2n + 1/2)\pi)$
ii. $I = (2n\pi, (2n + 1)\pi)$
iii. $I = ((2n + 1/2)\pi, (2n + 3/2)\pi)$
iv. $I = ((2n + 1/2)\pi, (2n + 3/2)\pi)$

for some integer $n \in \mathbb{Z}$.

In order, the images under $f$ of $I$ are

i. $f(I) = H_{x>0} \cap S^1$
ii. $f(I) = H_{y>0} \cap S^1$
iii. $f(I) = H_{x<0} \cap S^1$
iv. $f(I) = H_{y<0} \cap S^1$

which in each case is an open set in $S^1$.

In order, the functions

i. $g((x, y)) = 2n + \arcsin(y)$
ii. $g((x, y)) = 2n + \arccos(x)$
iii. $g((x, y)) = 2n + 1 + \arcsin(y)$
iv. $g((x, y)) = 2n + 2 + \arccos(x)$
give a continuous inverse to the function \( f_I : I \to f(I) \subset S^1 \). Therefore the image of \((a, b) \subset I\) is an open subset of \( f(I) \), which in turn is open in \( S^1 \), and so \( f((a, b)) \) is open, as required.

(d) The scaling function \( \mathbb{R} \to \mathbb{R} \) given by \( \theta \mapsto 2\pi \theta \) is a homeomorphism. We deduce that the function \( \mathbb{R} \to S^1 \) given by \( x \mapsto (\cos 2\pi \theta, \sin 2\pi \theta) \) is an open map.

It is sufficient to prove that the map \( h \) is open, since we already know it is bijective and continuous (from class). To prove it is open, it is sufficient to prove that for each point \( x \in [0,1]/\{0,1\} \), there is some local base at \( x \) that \( h \) maps to an open set in \( S^1 \). For all points \( x \in (0,1) \), this follows immediately from the previous parts of this question. The remaining case is the point \( * \) obtained by identifying \( 0 \sim 1 \). The open neighbourhoods of \( * \) are precisely the images of open neighbourhoods of \( \{0,1\} \) in \( [0,1] \). A local base can therefore be written down for \( * \): for instance the sets \( \{* \cup (0,\epsilon) \cup (1-\epsilon,1)\} \) for \( 0 < \epsilon < 1/4 \). The image under \( h \) of such a set is again open by reference to Part 2.

\[ \Box \]

4. We know that sequences are not sufficient in general topological spaces to determine when \( x \in \bar{A} \). This question shows that a kind of generalized sequence is sufficient. A directed set \( D \) is a nonempty set equipped with a relation \( \leq \) satisfying:

(a) \( x \leq x \) for all \( x \in D \).

(b) if \( x \leq y \) and \( y \leq z \) then \( x \leq z \)

(c) for all \( x, y \) there exists some \( w \) such that \( x \leq w \) and \( y \leq w \).

Given a directed set \( D \), define set \( D \cup \{\infty\} \) by adding a disjoint point “\( \infty \)”. Put a topology on \( D \cup \{\infty\} \) by declaring the following sets to be open:

- All subsets of \( D \)
- All subsets of \( D \cup \{\infty\} \) containing a set of the form \( S_d = \{x \in D \mid x \geq d\} \cup \{\infty\} \) where \( d \in D \).

You do not have to verify that this is a topology.

Let \( X \) be a topological space. Any function \( f : D \to X \) is called a net in \( X \). The net is said to converge to \( x \in X \) if there exists a continuous function \( \tilde{f} : D \cup \{\infty\} \to X \) such that \( \tilde{f}(\infty) = x \) and \( \tilde{f}|_D = f \). Suppose \( A \subset X \) is a subset and \( x \in \bar{A} \). Produce a directed set \( D \) and a net \( f : D \to X \) with image in \( A \) that converges to \( x \).

If \( x \in A \), we can set \( D = * \) and define both \( f : D \to X \) and \( \tilde{f} : D \cup \{\infty\} \to X \) to be constant functions with value \( x \). Therefore we may assume that \( x \not\in A \).

Let \( D \) denote the set of open sets \( U \ni x \) in \( X \). We verify that this is a directed subset under the relation \( U \leq V \) if \( V \subset U \). Clearly this relation is reflexive and transitive. If \( U \ni x \) and \( W \ni x \) are open sets, then \( U \cap W \ni x \) is an open set as well. Both \( U \leq U \cap W \) and \( W \leq U \cap W \). Therefore \( D \) is a directed set.

Now define a function \( f : D \to X \) as follows: for each \( U \ni x \) there is some \( a_U \in A \cap U \). This is because \( x \in \bar{A} \), and so every open neighbourhood of \( x \) intersects \( A \). Our hypothesis that \( x \notin A \) ensures that \( x \neq a_U \).
Let $f(U) = a_U^2$. Extend this to $\tilde{f} : D \cup \{\infty\} \to X$ by setting $\tilde{f}(\infty) = x$. By construction, the image of $f$ is in $A$. We claim that $\tilde{f}$ is continuous.

Let $U \subseteq X$ be an open set. If $x \notin U$, then $\tilde{f}^{-1}(U) \neq \infty$. Since any subset of $D$ is open in $D \cup \{\infty\}$, in particular, $\tilde{f}^{-1}(U)$ is open. If $x \in U$, then consider the element $U \in D$. We claim that $S_U \subseteq \tilde{f}^{-1}(U)$. Let $V \supseteq U$ in $D$, i.e., $V \subseteq U$ as open subsets of $X$. Then $a_V \in VU$, so $a_V \in U$, so that $\tilde{f}(V) = a_v \in U$. Equivalently, $V \in \tilde{f}^{-1}(U)$. We deduce the containment $S_U \subseteq \tilde{f}^{-1}(U)$, so that $\tilde{f}^{-1}(U)$ is open.

It follows that $f : D \to X$ converges to $x$. □

5. Let $\mathbb{R}^\mathbb{N}$ denote the set $\prod_{i=1}^{\infty} \mathbb{R}$ (i.e. the set of all sequences $(r_1, r_2, \ldots)$ of real numbers), and let $c_{00}$ denote the subset consisting of those sequences $(r_1, r_2, \ldots)$ that are eventually 0, that is, for which there exists $N \in \mathbb{N}$ such that $r_i = 0$ for $i > N$.

(a) Give $\mathbb{R}^\mathbb{N}$ the product topology. Show that $\overline{c_{00}} = \mathbb{R}^\mathbb{N}$.

(b) Give $\mathbb{R}^\mathbb{N}$ the box topology. What is $\overline{c_{00}}$?

(a) Let $x = (x_1, x_2, \ldots) \in \mathbb{R}^\mathbb{N}$ be an element, and let $V \ni x$ be a neighbourhood of $x$. Then $V$ contains a basis element containing $x$, i.e. a set of the form $U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R} \times \mathbb{R} \times \cdots$, where each $U_i$ is open in $\mathbb{R}$. Since $x_i \in U_i$, no $U_i$ can be empty. We may choose an element $r = (r_1, r_2, \ldots, r_n, 0, 0, 0, \ldots)$ where $r_i \in U_i$. Then $r \in V \cap c_{00}$, and so every open neighbourhood of $x$ has nonempty intersection with $c_{00}$, therefore $x \in \overline{c_{00}}$.

(b) Suppose $x \notin c_{00}$. Construct an open neighbourhood as follows: if $x_i = 0$, then let $U_i = \mathbb{R}$. If $x_i \neq 0$, then let $U_i \subseteq \mathbb{R}$ be an open neighbourhood of $x_i$ that does not contain 0. Since $x \notin c_{00}$, we must have $0 \notin U_i$ for infinitely many values of $i$. Then let $V = U_1 \times U_2 \times \cdots$, which is an open neighbourhood of $x$ in the box topology. This neighbourhood does not contain any sequence that is eventually 0 since there are infinitely many $U_i \neq 0$, so $x \notin \overline{c_{00}}$. It follows that $c_{00} = \overline{c_{00}}$. □

Note for pedants: we have used the axiom of choice here.