1. Give $\mathbb{N}$ the cofinite topology, where the nonempty open sets are those sets containing all but finitely many elements. Give $\mathbb{R}$ the usual topology. Describe, with proof, the continuous functions $f : \mathbb{N} \to \mathbb{R}$.

2. Here $A$, $B$ and the family $\{A_i\}_{i \in I}$ are subsets of a topological space $X$. In each case, prove the identity given is true, or give a counterexample to show it is false. For producing counterexamples, it may be helpful to know that in a metric topology, any singleton subset $\{x\}$ is closed, since for $y \in X \setminus \{x\}$ we have $B(y, d(x, y)) \subset X \setminus \{x\}$.

   (a) $A \cup B = \overline{A \cup B}$,
   
   (b) $\bigcup_{i \in I} A_i = \overline{\bigcup_{i \in I} A_i}$,
   
   (c) If $A$ is open, then $\text{Int} A = A$,
   
   (d) $\partial A = \emptyset$ if and only if $A$ is both closed and open.

Moreover, prove that

   (e) $\partial A$ is the set of points $x \in X$ such that for all open $U \ni x$, both $U \cap A \neq \emptyset$ and $U \cap (X \setminus A) \neq \emptyset$.

3. Let $\mathbb{R}^n$ be given the usual (metric) topology, and let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be given the subspace topology as a subset of $\mathbb{R}^2$. Consider the function $f : \mathbb{R} \to S^1$ given by $f(\vartheta) = (\cos \vartheta, \sin \vartheta)$. You may assume $f$ is continuous. Let $H = H_{x>0}$ denote the subset of $S^1$ consisting of points where $x > 0$. Define $H_{x<0}$, $H_{y>0}$ and $H_{y<0}$ similarly.

   (a) Give an open interval $I \subset \mathbb{R}$ and a continuous function $g : \{(x, y) \in \mathbb{R}^2 \mid -1 < y < 1\} \to \mathbb{R}$ such that $f(I) = H$ and $g(H) = I$, and that the restricted map $f|_I : I \to H$ is a homeomorphism

1 This is an abuse of notation: strictly, by $f|_I$ we mean the unique function $\phi : I \to H$ such that the composite of $\phi$ with the inclusion $H \subset S^1$ agrees with the composite of the inclusion $I \subset \mathbb{R}$ with $f : \mathbb{R} \to S^1$.

4. Let $\{(A_i, \tau_i)\}_{i \in I}$ be a family of topological spaces. Let $A = \prod_{i \in I} A_i$ and let $\tau$ denote the product topology.
Let $U$ be a nonempty open set and denote $\text{im}(\pi_i|_U)$ by $\pi_i(U)$. Note that

   $\pi_i(U) = A_i$

for all but finitely many values of $i \in I$ (we say for almost all values of $i$). This can be proved by proving it for the elements of the basis consisting of finite intersections of sets of the form $\pi_i^{-1}(U)$.
(a) Define a topology, \( \beta \), on \( A = \prod_{i \in I} A_i \) as the topology generated by sets of the form

\[
\bigcap_{i \in I} \pi^{-1}_i(V_i)
\]

where \( V_i \) is open in \( A_i \) for all values of \( i \in I \). This is the box topology. Prove that the box topology is finer than the product topology.

(b) Prove that if \( I \) is finite, the box topology and product topology are the same.

(c) Prove that if \( I = \mathbb{N} \) and \( A_i = \mathbb{R} \) with the usual topology that the box and product topologies are not the same. It may be helpful here, and subsequently, to understand this product \( \prod_{i=1}^{\infty} \mathbb{R} \) as the set of all infinite sequences of real numbers.

5. Read at least the first three sections of the supplementary notes on \( p \)-norms (the fourth section will be required for a later homework assignment).

(a) Show that for a given vector \( x \in \mathbb{R}^n \), the function \( p \mapsto \|x\|_p \) is (weakly) decreasing on \( p \in [1, \infty) \).

(b) Show that \( \lim_{p \to \infty} \|x\|_p = \|x\|_\infty \).