1. Read the supplementary notes on $p$-norms.
   (a) Show that for a given vector $x \in \mathbb{R}^n$, the function $p \mapsto \|x\|_p$ is (weakly) decreasing on $p \in [1, \infty]$.
   (b) Show that $\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$.

2. Again, refer to the supplementary notes on $p$-norms.
   (a) Suppose $\{(X_1, d_1), \ldots, (X_n, d_n)\}$ is a finite set of metric spaces and write $X = \prod_{i=1}^n X_i$. Let $p \in [0, \infty]$. Define a product $p$-metric on $X$ as follows. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are elements of $X$, then let
   \[ d(x, y) = \|(d_1(x_1, y_1), d_2(x_2, y_2), \ldots, d_n(x_n, y_n))\|_p. \]
   Prove that $d$ is a metric on $X$.
   (b) Prove that $d_p(x, y) = \|x - y\|_p$ defines a metric on $\mathbb{R}^n$ for any $p \in [1, \infty]$. You can do this directly, or apply the previous part to an $n$-fold product of $(\mathbb{R}, |\cdot|)$.
   (c) Let $p, q \in [1, \infty]$. Show that the metric topology associated to $d_p$ and to $d_q$ on $\mathbb{R}^n$ agree. It may be helpful to refer to Lemma 20.2 of Munkres' Topology.

3. Let $\mathcal{C}$ be a category and let $f : A \to B$ be a morphism in $\mathcal{C}$. We say $f$ is an epimorphism if it has the property that whenever $g_1 : B \to X$ and $g_2 : B \to X$ are morphisms such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$. Describe, with proof, the epimorphisms in the category of sets.

4. Give $\mathbb{N}$ the cofinite topology, where the nonempty open sets are those sets containing all but finitely many elements. Give $\mathbb{R}$ the usual topology. Describe, with proof, the continuous functions $f : \mathbb{N} \to \mathbb{R}$.

5. Let $\{(A_i, \tau_i)\}_{i \in I}$ be a family of topological spaces. Let $A = \prod_{i \in I} A_i$ and let $\tau$ denote the product topology. Let $U$ be a nonempty open set and denote $\text{im}(\pi_i|_U)$ by $\pi_i(U)$. Note that
   \[ \pi_i(U) = A_i \]
   for all but finitely many values of $i \in I$ (we say for almost all values of $i$). This can be proved by proving it for the elements of the basis consisting of finite intersections of sets of the form $\pi_i^{-1}(U)$.
   (a) Define a topology, $\beta$, on $A = \prod_{i \in I} A_i$ as the topology generated by sets of the form
   \[ \bigcap_{i \in I} \pi_i^{-1}(V_i) \]
   where $V_i$ is open in $A_i$ for all values of $i \in I$. This is the box topology. Prove that the box topology is finer than the product topology.
(b) Prove that if $I$ is finite, the box topology and product topology are the same.

(c) Prove that if $I = \mathbb{N}$ and $A_i = \mathbb{R}$ with the usual topology that the box and product topologies are not the same. It may be helpful here, and subsequently, to understand this product $\prod_{i=1}^{\infty} \mathbb{R}$ as the set of all infinite sequences of real numbers.

6. Let $\mathbb{R}^\mathbb{N}$ denote the set $\prod_{i=1}^{\infty} \mathbb{R}$ (i.e. the set of all sequences $(r_1, r_2, \ldots)$ of real numbers), and let $c_{00}$ denote the subset consisting of those sequences $(r_1, r_2, \ldots)$ that are eventually 0, that is, for which there exists $N \in \mathbb{N}$ such that $r_i = 0$ for $i > N$.

(a) Read Theorem 19.1 of Munkres’ topology.

(b) Give $\mathbb{R}^\mathbb{N}$ the product topology. Show that $c_{00} = \mathbb{R}^\mathbb{N}$.

(c) Give $\mathbb{R}^\mathbb{N}$ the box topology. What is $c_{00}$?

\begin{footnotesize}
\textsuperscript{1}In a previous version of this homework, there was a mistake in the definition of the box topology. This has been fixed.

\textsuperscript{2}After having written this homework, I realized I would want the notation $c_0$ for a different set, the set of sequences converging to 0. So the notation here has been changed to $c_{00}$.
\end{footnotesize}