1. Let \( \mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \) and \( \mathbf{z} = (z_1, \ldots, z_n) \) be elements of \( \mathbb{R}^n \). Let \( \| \mathbf{x} \|_2 = \sqrt{\sum_{i=1}^n x_i^2} \).

(a) Show that
\[
\sum_{i=1}^n \sum_{j=1}^n x_i^2 y_j^2 \geq \sum_{i=1}^n x_i y_i y_j.
\]
Show that equality holds if and only if \( \{ \mathbf{x}, \mathbf{y} \} \) is linearly dependent. If you are stuck, try the cases \( n = 2 \) and \( n = 3 \).

(b) Show that
\[
\| \mathbf{x} \|_2 \cdot \| \mathbf{y} \|_2 \geq \left| \sum_{i=1}^n x_i y_i \right|
\]
with equality if and only if \( \{ \mathbf{x}, \mathbf{y} \} \) is linearly dependent. This is called the Cauchy-Schwarz inequality.

(c) Show that
\[
\| \mathbf{x} \|_2 + \| \mathbf{y} \|_2 \geq \| \mathbf{x} + \mathbf{y} \|_2.
\]
(d) Conclude that \( \| \mathbf{x} - \mathbf{z} \|_2 + \| \mathbf{z} - \mathbf{y} \|_2 \geq \| \mathbf{x} - \mathbf{y} \|_2 \). This is the triangle inequality for the usual metric on \( \mathbb{R}^n \).

2. Let \( \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and similarly for \( \mathbf{y} \).

(a) Define a function \( d_1 : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty) \) by \( d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i| \). Prove that \( d_1 \) is a metric.

(b) Define a function \( d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty) \) by \( d_\infty(\mathbf{x}, \mathbf{y}) = \sup_{i \in \{1, \ldots, n\}} |x_i - y_i| \). Prove that \( d_\infty \) is a metric.

(c) Let \( n = 2 \). Illustrate the unit balls \( B((0,0), 1) \) for each of the metrics \( d_1, d_2 \) and \( d_\infty \).

3.

(a) Suppose \((X, d)\) is a metric space where \( X \) is not empty. Let \( x \in X \) be an element. Prove the following are equivalent:

i. There exists some \( R \in \mathbb{R} \) such that \( B(x, R) = X \).

ii. There exists some \( S \in \mathbb{R} \) such that \( d(y, z) < S \) for all \( y, z \in S \).

We call metric spaces with either of these equivalent properties \emph{bounded}, and we say the metric is bounded.

(b) Suppose \((X, d)\) is a metric space. Prove that the function
\[
s(x, y) = \frac{d(x, y)}{1 + d(x, y)}
\]
is a bounded metric on \( X \). It may be helpful to show that \( f(z) = z/(1 + z) \) is an increasing function on \([0, \infty)\).
(c) In the notation of the previous part, prove that the open balls in \((X, d)\) and in \((X, s)\) coincide. Therefore they induce the same metric topologies.

4. Suppose \(X, Y, Z\) are topological spaces and \(f : X \to Y\) and \(g : Y \to Z\) are continuous functions. Prove that \(g \circ f\) is continuous.

5. Let \(p\) be a prime number. Let \(q \in \mathbb{Q} - \{0\}\) be a nonzero rational number. Then \(q\) has a presentation in the following form

\[q = p^s \frac{a}{b}\]

where \(s \in \mathbb{Z}\) is an integer, \(a \in \mathbb{Z}\) is an integer, \(b \in \mathbb{N}\) is a natural number, and neither \(a\) nor \(b\) is divisible by \(p\). We will call such a presentation \(p\)-good.

(a) A number \(q\) may have several \(p\)-good presentations. Prove that for any two such presentations, the number \(s\) is the same.

We define a function \(\nu_p : \mathbb{Q} - \{0\} \to \mathbb{Z}\) by \(\nu_p(q) = s\) when \(q = p^s(a/b)\) is a \(p\)-good presentation of \(q\).

Extend this to \(\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}\) by declaring \(\nu_p(0) = \infty\).

For \(q \in \mathbb{Q}\), define \(|q|_p = \exp(-\nu_p(q))\) when \(q \neq 0\), with the 'obvious' definition of

\[|0|_p = \lim_{t \to \infty} \exp(-t) = 0.\]

(b) Verify that \(|qr|_p = |q|_p|r|_p\) for any \(q, r \in \mathbb{Q}\).

(c) Verify that \(|q + r|_p \leq \max\{|q|_p, |r|_p\}\) for any \(q, r \in \mathbb{Q}\).

We define

\[d_p : \mathbb{Q} \times \mathbb{Q} \to [0, \infty)\]

by

\[d_p(q, r) = |q - r|_p.\]

(d) Prove that \(d_p(q, r)\) satisfies the following inequality: If \(q, r, t \in \mathbb{Q}\), then \(d_p(q, r) \leq \max\{d_p(q, t), d_p(t, r)\}\).

(e) Prove that \(d_p\) is a metric. The metric topology for \(d_p\) is called the \(p\)-adic topology.

(f) Describe \(B_p(0, 1) \cap \mathbb{Z}\).