1. Recall the concept of “ambient isotopy” from a previous homework. Write $D^n \subset \mathbb{R}^n$ for the closed unit disk, and $B^n$ for the open unit ball, both centred at 0. Recall that $\partial D^n = S^{n-1}$.

(a) Suppose $q$ is a point in $\mathbb{R}^n$. Verify that $H(x, t) = x - tq$ gives an ambient isotopy from $\{q\}$ to $\{0\}$, the origin, in $\mathbb{R}^n$.

(b) We may use the homeomorphism $\phi : \mathbb{R}^n \to B^n$ and its inverse $\phi^{-1}$ given by the formulas

$$
\phi(x) = \frac{x}{1 + \|x\|}, \quad \phi^{-1}(y) = \frac{y}{1 - \|y\|}
$$

to view $D^n$ as a compactification of $\mathbb{R}^n$. Write $p = \phi(q)$. Show that the ambient isotopy in the previous part of the question extends to an ambient isotopy $\tilde{H} : D^n \times I \to D^n$ taking $\{p\}$ to $\{0\}$, relative to $S^{n-1}$.

In this solution we write $H_t(x)$ for $H(x, t)$.

(a) This is elementary. $H$ is continuous, $H_t$ is a homeomorphism for all $t$, since one can write down the inverse $x \mapsto x + t q$, $H_0$ is the identity and $H_1(q) = 0$.

(b) To define $\tilde{H}|_{B^n}$, use $\tilde{H}_t = \phi \circ H_t \circ \phi^{-1}$. This has formula (after clearing denominators):

$$
\tilde{H}_t(x) = \frac{(1 - \|p\|)x - (1 - \|x\|)tp}{(1 - \|x\|)(1 - \|p\|) + \|1 - \|p\||x - (1 - \|x\|)tp}
$$

We note that if $\|x\| = 1$, then the formula above makes sense. Therefore we take this formula to be our definition of $\tilde{H} : D^n \times I \to D^n$.

About the formula for $\tilde{H}_t(x)$, we verify the following:

- It agrees with $\phi \circ H_t \circ \phi^{-1}$ on $B^n$.
- Since $\|p\| < 1$, the denominator is positive for all $x \in D^2$: if $\|x\| < 1$, then the first summand in the denominator is positive and the second is nonnegative, while if $\|x\| = 1$, the second summand is positive and the first is 0.
- Since algebraic operations and $\|\cdot\|$ are continuous (where defined), it follows that $\tilde{H}$ is a continuous function.
- When $t = 0$, the formula collapses to give $\tilde{H}(x, 0) = x$ for all $x$.
- When $\|x\| = 1$, the formula also collapses to give $\tilde{H}(x, t) = x$ for all $t$.
• When $t = 1$, we see $\tilde{H}(p, 1) = 0$.

It remains to show that $\tilde{H}_t$ is a homeomorphism for all $t \in I$. Since $\tilde{H}_t$ is continuous and has source and target $D^n$ (which is compact and Hausdorff) it suffices to prove that $\tilde{H}_t$ is bijective for all $t$. But $\tilde{H}_t|_{B^n} : B^n \to B^n$ is a bijection, since it agrees with the homeomorphism $\phi \circ H_t \circ \phi^{-1}$, and $\tilde{H}_t|_{\partial D^n} : \partial D^n \to \partial D^n$ is the identity, so this too is a bijection. This shows that $\tilde{H}_t$ is bijective, as required.

2. Let $D^2$ denote the closed unit disk in $\mathbb{R}^2$, that is $D^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$.

(a) Prove that there does not exist a retraction for the inclusion $i : S^1 \to D^2$ (i.e., there is no map $r : D^2 \to S^1$ such that $r \circ i = \text{id}_{S^1}$).

(b) Suppose $f : D^2 \to D^2$ is a continuous map. Prove that there is some $x \in D^2$ such that $x = f(x)$. Hint: consider the rays starting at $f(x)$ and passing through $x$.

(a) Let $s \in S^1$ be a basepoint. Suppose for the sake of contradiction that $r$ is a retraction. Then $r \circ i = \text{id}_{S^1}$. Applying the functor $\pi_1$ gives us a commutative diagram

$$
\pi_1(S^1, s) \xrightarrow{i_*} \pi_1(D^2, s) \xrightarrow{r_*} \pi_1(S^1, s)
$$

where the first and last two groups are infinite cyclic, but the middle group is trivial. This is impossible.

(b) We suppose $f$ exists and we will derive a contradiction. For each $x$, let $R$ denote the ray starting at $f(x)$ and passing through $x$—this is uniquely defined because $f(x) \neq x$. Let $p(x)$ denote the point where $R$ intersects $S^1$. One may calculate

$$
p(x) = t_x f(x) + (1 - t_x)x
$$

where $t_x$ is the nonnegative root of the quadratic equation $\|tf(x) + (1 - t)x\|^2 = 1$. Since the coefficients of this equation depend continuously on $t$, the root $t_x$ depends continuously on $x$ and therefore so does $p(x)$.

It is immediate that $p(x) = x$ if $x \in S^1$.

But now $p : D^2 \to S^1$ is a continuous function such that $p(x) = x$ for all $x \in S^1$, i.e., it gives a retraction of $D^2$ onto $S^1$. This is a contradiction.
3. Consider the following three subspaces of $\mathbb{R}^3$:

- $X_1 = \{(0, 0, z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\}$;
- $X_2 = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$;
- $X_3 = \{(0, y, z) \in \mathbb{R}^3 \mid (y - 1)^2 + z^2 = 1\}$.

Choose a basepoint $p \in \mathbb{R}^3 \setminus (X_1 \cup X_2 \cup X_3)$.

(a) Calculate $\pi_1(\mathbb{R}^3 \setminus (X_1 \cup X_2), p)$.

(b) Consider the picture in Figure 1. Express $[\gamma]$ in terms of $[\alpha]$ and $[\beta]$ in $\pi_1(\mathbb{R}^3 \setminus (X_1 \cup X_2), p)$. Proof-by-picture is sufficient, provided it’s clear.

(c) Calculate $\pi_1(\mathbb{R}^3 \setminus (X_2 \cup X_3), p)$.

(a) The space $\mathbb{R}^3 \setminus (X_1 \cup X_2)$ admits $T$ as a deformation retract, where $T$ is the standard embedding of the torus. This is depicted in Figure 2. The deformation retract is described in Figure 3, where a vertical slice of $\mathbb{R}^3 \setminus (X_1 \cup X_2)$ is demonstrated. The retracting homotopy is radially symmetric.

Since $\pi_1(S^1 \times S^1, x) \cong \mathbb{Z} \times \mathbb{Z}$, it follows that $\pi_1(\mathbb{R}^3 \setminus (X_1 \cup X_2), x)$ is also free abelian of rank 2.

(b) The class of $\alpha$ is equal to that of $\gamma$. To see this, enlarge the loop $\alpha$ until it is very large (larger than the horizontal circle) and then allow $\alpha$ to slide downwards until it is below the $xy$-plane. Then shrink it until it coincides with $\gamma$. 

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(c) We use van Kampen’s theorem with two open sets $U$ and $V$ as indicated in Figure 4. Of these, $U$ is homeomorphic to $\mathbb{R}^3 \setminus (X_1 \cup X_2)$, so we know its fundamental group: it is free abelian generated by two symbols $\langle \alpha, \beta | \alpha \beta = \beta \alpha \rangle$, the space $V$ is homeomorphic to $\mathbb{R}^3 \setminus X_1 \cong S^1$, so the fundamental group is an infinite cyclic group, $\pi_1(V, x) \cong \langle \delta \rangle$. The intersection $U \cap V$ is homotopy equivalent to $S^1 \lor S^1$, so $\pi_1(U \cap V, x) \cong \langle \epsilon, \zeta \rangle$. We may take $\epsilon$ to be the class of the loop drawn in pink in Figure 4 and $\zeta$ to be the class of the curve drawn in orange.

We also identify the homomorphisms induced by inclusion

$$\phi : \pi_1(U \cap V, x) \to \pi_1(U, x) \quad \phi(\epsilon) = \alpha = \phi(\zeta)$$

$$\psi : \pi_1(U \cap V, x) \to \pi_1(U, x) \quad \phi(\epsilon) = \delta = \phi(\zeta)$$

The first of these is essentially the content of part (b), and the second is obvious.

By the Van Kampen theorem, $\pi_1(\mathbb{R}^3 \setminus (X_2 \cup X_3), x)$ is the amalgamated product

$$\pi_1(\mathbb{R}^3 \setminus (X_2 \cup X_3), x) = \langle \alpha, \beta, \delta | \alpha \beta = \beta \alpha, \alpha = \delta \rangle$$

which simplifies immediately to

$$\pi_1(\mathbb{R}^3 \setminus (X_2 \cup X_3), x) = \langle \alpha, \beta | \alpha \beta = \beta \alpha \rangle.$$
Figure 3: A vertical slice of the deformation retract $S^1 \times S^1 \hookrightarrow \mathbb{R}^3 \setminus (X_1 \cup X_2)$, with the retracting homotopy indicated in pink.

**Note:** One can also observe that $\mathbb{R}^3 \setminus (X_2 \cup X_3)$ admits $S^2 \vee (S^1 \times S^1)$ as a deformation retract. This gives another way to get the same answer. The embedding of two linked circles in $\mathbb{R}^3$ that appears in this question is called the *Hopf link.*
Figure 4: An open cover for applying Van Kampen’s Theorem to $R^3 \setminus (X_2 \cup X_3)$